Geometric Sum

Part a. Determine a closed-form expression for the following sum:

$$S_N = \sum_{n=0}^{N-1} a^n$$

Hint: Compare S_N to aS_N.

Is your closed-form expression valid for all a and N? If not, determine alternative expressions for these (and any other) edge cases.

$$S_N = \sum_{n=0}^{N-1} a^n = 1 + a + a^2 + \cdots + a^{N-1}$$

$$aS_N = \sum_{n=0}^{N-1} a^n = a + a^2 + \cdots + a^{N-1} + a^N$$

$$S_N - aS_N = (1-a)S_N = 1-a^N$$

If $a \neq 1$ then

$$S_N = \frac{1-a^N}{1-a}$$

If $a = 1$ then

$$S_N = N$$

so the final solution is

$$S_N = \begin{cases} \frac{1-a^N}{1-a} & \text{if } a \neq 1 \\ N & \text{otherwise} \end{cases}$$

This expression holds for all a and for all $N > 0$.

Part b. Determine a closed-form expression for the following sum:

$$S_\infty = \sum_{n=0}^{\infty} a^n$$

Is your closed-form expression valid for all a? Explain.

Take the limit of the answer for S_N as $N \to \infty$. Then

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1-a^N}{1-a} = \frac{1}{1-a}$$

provided $a \neq 1$ (so that the denominator is not zero) and $|a| < 1$ (so that $\lim_{N\to\infty} a^N = 0$). Both of these conditions are met if $|a| < 1$. If $|a| \geq 1$, then S_N diverges.

Part c. Determine a closed-form expression for the following sum (with infinitely many terms):

$$S_d = 1 + 2a + 3a^2 + 4a^3 + 5a^4 + \cdots$$

Hint: Differentiate.

Is your closed-form expression valid for all a? Explain.

Equate the series expression for S_∞ with the corresponding closed-form:

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

and then differentiate both sides with respect to a.

$$\frac{d}{da}\sum_{n=0}^{\infty} a^n = \sum_{n=0}^{\infty}\frac{d}{da}a^n = \sum_{n=0}^{\infty} na^{n-1} = \frac{d}{da}\left(\frac{1}{1-a}\right) = \frac{1}{(1-a)^2}$$

But

$$= \sum_{n=0}^{\infty} na^{n-1} = 1 + 2a + 3a^2 + 4a^3 + 5a^4 + \cdots$$

Therefore

$$1 + 2a + 3a^2 + 4a^3 + 5a^4 + \cdots = \frac{1}{(1-a)^2}$$

As in part b, this expression only holds if $|a| < 1$. If $|a| \geq 1$, then this series diverges.