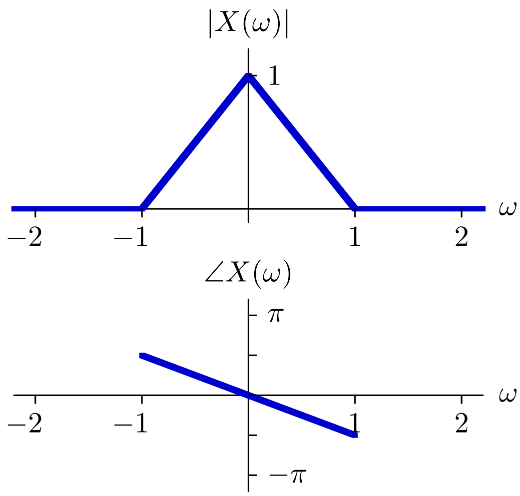
Fourier Match

The magnitude and angle of the Fourier transform of a signal $x(t)$ are given in the following plots.



Five signals are derived from $x(t)$:

$$x_1(t) = \frac{dx(t)}{dt}$$

$$x_2(t) = (x * x)(t)$$

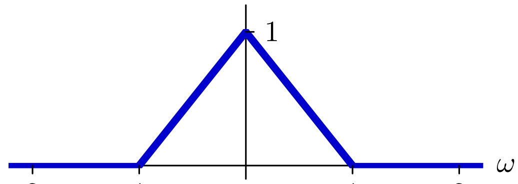
$$x_3(t) = x\left(t - \frac{\pi}{2}\right)$$

$$x_4(t) = x(2t)$$

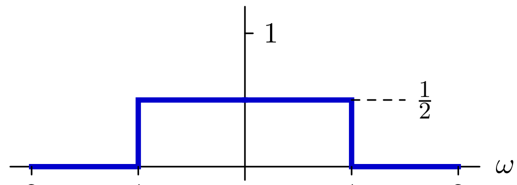
$$x_5(t) = x^2(t)$$

Seven magnitude plots (M1-M7) and seven angle plots (A1-A7) are shown on the following page. Determine which of these plots is associated with each of the derived signals.

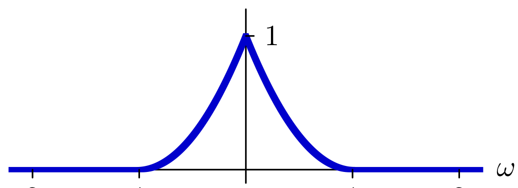
M1



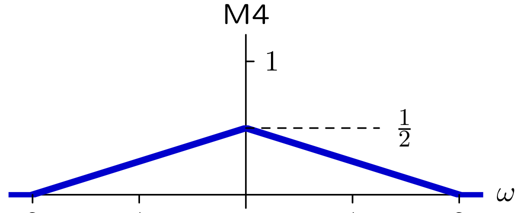
M2



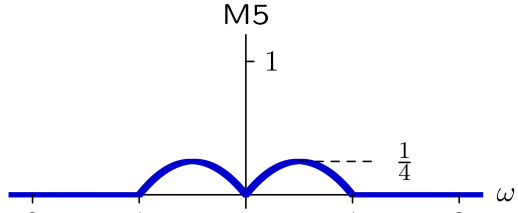
M3



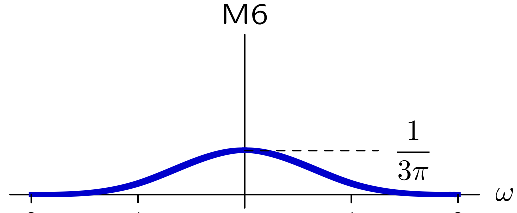
M4



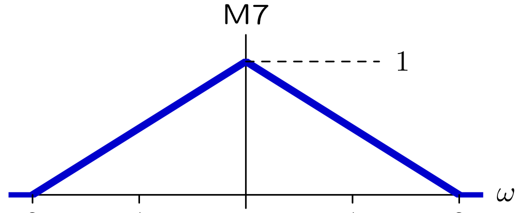
M5



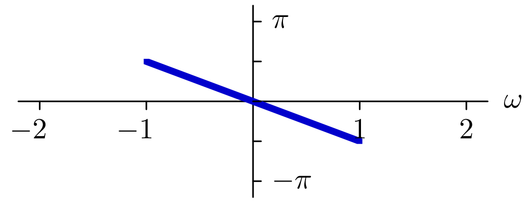
M6



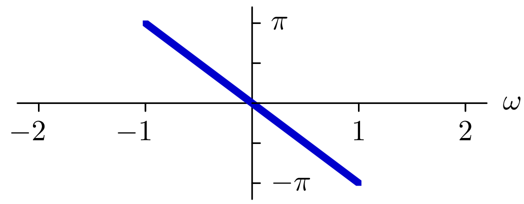
M7



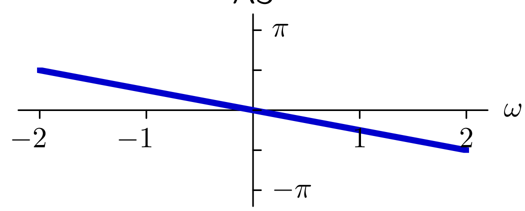
A1



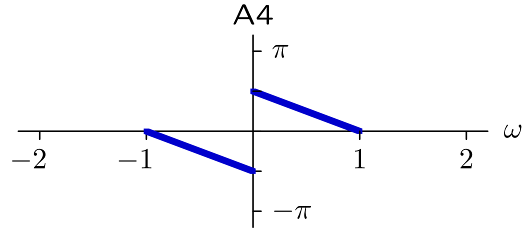
A2



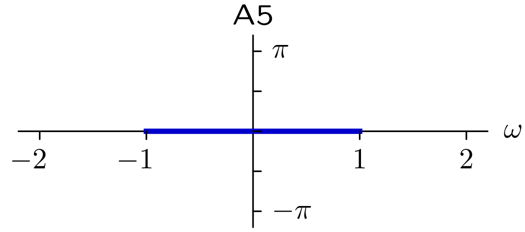
A3



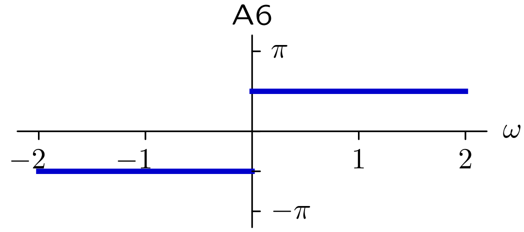
A4



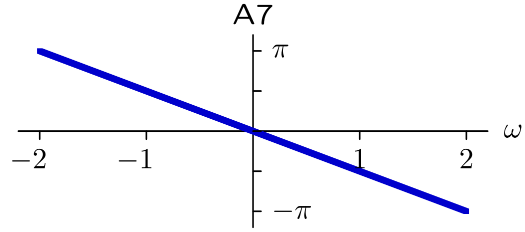
A5



A6



A7



Part a. Which plot shows the magnitude of $\frac{dx(t)}{dt}$?

If a signal is differentiated in time, its Fourier transform is multiplied by $j\omega$. Thus the magnitude is multipled by $|\omega|$. The result is symmetric about $\omega = 0$. The shape is parabolic over the interval $(0, 1)$, reaching a peak value of $\frac{1}{4}$ at $\omega = \pm\frac{1}{2}$. Therefore the answer is M5.

Part b. Which plot shows the angle of $\frac{dx(t)}{dt}$?

If a signal is differentiated in time, its Fourier transform is multiplied by $j\omega$. Thus the angle is increased by $\frac{\pi}{2}$ (since $j = e^{j\frac{\pi}{2}}$) if $\omega > 0$ and decreased by $\frac{\pi}{2}$ if $\omega < 0$. The result is an odd function of ω (as the angle must be if the time function is real), tapering from $\frac{\pi}{2}$ at $\omega = 0$ to 0 at $\omega = 1$. Therefore the answer is A4.

Part c. Which plot shows the magnitude of $(x * x)(t)$?

If a signal is convolved with itself, the magnitude of the Fourier transform is squared. The result is 1 at $\omega = 0$ and 0 for $|\omega| > 1$. The square of a number that is between 0 and 1 is less than the number. For example, the result at $\omega = \frac{1}{2}$ is $\frac{1}{4}$. Therefore the answer is M3.

Part d. Which plot shows the angle of $(x * x)(t)$?

If a signal is convolved with itself, the Fourier transform is squared. Thus the angle is doubled. Therefore the answer is A2.

Part e. Which plot shows the magnitude of $x\left(t - \frac{\pi}{2}\right)$?

Delaying in time alters the angle of the Fourier transform but not the magnitude. Therefore the answer is M1.

Part f. Which plot shows the angle of $x\left(t - \frac{\pi}{2}\right)$?

Delaying a signal by $\frac{\pi}{2}$ seconds multiples the transform by $e^{-j\frac{\pi}{2}\omega}$. This decreases the angle by $\frac{\pi}{2}\omega$ for $\omega > 0$ and increases the angle by $\frac{\pi}{2}|\omega|$ for $\omega < 0$. Therefore the answer is A2.

Part g. Which plot shows the magnitude of $x(2t)$?

Compressing time stretches frequency. Therefore the answer is M4.

Part h. Which plot shows the angle of $x(2t)$?

Compressing time stretches frequency. Therefore the answer is A3.

Part i. Which plot shows the magnitude of $x^2(t)$?

If a signal is squared, its Fourier transform is convolved with itself. Convolution of a triangle of width 2 with itself produces a result of width 4. Between $\omega = -2$ and $\omega = -1$, the magnitude grows with the square of ω. Between -1 and 0, it grows more slowly than the square. And the result is symmetric in ω. Therefore the answer is M6.

Part j. Which plot shows the angle of $x^2(t)$?

Squaring a signal corresponds to convolving its Fourier transform with itself. Let $X(\omega)$ represent the original transform. Then

$$(X * X)(\omega) = \frac{1}{2\pi}\int \Big|X(\lambda)\Big| \times e^{j\angle X(\lambda)} \times \Big|X(\omega-\lambda)\Big| \times e^{j\angle X(\omega-\lambda)} d\lambda$$

We can see from the original graph that $\angle X(\omega) = -\frac{\pi}{2}\omega$, which we can substitute in to the above equation:

$$\begin{aligned}
(X * X)(\omega) &= \frac{1}{2\pi}\int \Big|X(\lambda)\Big| \times e^{j\angle X(\lambda)} \times \Big|X(\omega-\lambda)\Big| \times e^{j\angle X(\omega-\lambda)} d\lambda \\
&= \frac{1}{2\pi}\int \Big|X(\lambda)\Big| \times e^{-j\frac{\pi}{2}\lambda} \times \Big|X(\omega-\lambda)\Big| \times e^{-j\frac{\pi}{2}(\omega-\lambda)} d\lambda \\
&= \frac{1}{2\pi}\int \Big|X(\lambda)\Big| \times \Big|X(\omega-\lambda)\Big| \times e^{-j\frac{\pi}{2}\lambda} \times e^{-j\frac{\pi}{2}(\omega-\lambda)} d\lambda \\
&= \frac{1}{2\pi}\int \Big|X(\lambda)\Big| \times \Big|X(\omega-\lambda)\Big| \times e^{-j\frac{\pi}{2}\omega} d\lambda \\
&= \frac{1}{2\pi} e^{-j\frac{\pi}{2}\omega}\int \Big|X(\lambda)\Big| \times \Big|X(\omega-\lambda)\Big| d\lambda
\end{aligned}$$

This form shows us that we find the magnitude of this new signal by convolving the magnitudes of the original signal, and that the angle of the new signal is $\angle X_5(\omega) = -\frac{\pi}{2}\omega$.

This must be true everywhere that $|X_5(\omega)|$ is nonzero, which is the range from -2 to 2 (for other ω values, the angle is undefined). Thus, the answer must be A7.