Exponentials and Geometrics

Part a. Find the Continuous-Time Fourier Transform of $f_1(t)$:

$$f_1(t) = e^{-|t|}$$

The signal $f_1(t)$ can be written as the sum of a left-hand part and a right-hand part:

$$f(t) = e^{-|t|} = e^{t}u(-t) + e^{-t}u(t)$$

Find the Fourier transform of the right-hand part.

$$F_r(\omega) = \int_{-\infty}^{\infty} e^{-|t|}e^{-j\omega t}dt \int_{0}^{\infty} e^{-t}e^{-j\omega t}dt = \left[\frac{e^{-(1+j\omega)t}}{-(1+j\omega)}\right]_0^{\infty} = \frac{1}{1+j\omega}$$

The left-hand part is a flipped version of the right-hand part, so the transform of the left-hand part is a flipped version of the transform of the right-hand part:

$$F_l(\omega) = F_r(-\omega) = \frac{1}{1-j\omega}$$

The transform of $f_1(t)$ is the sum of the transforms of the left-hand and right-hand parts:

$$F(\omega) = F_l(\omega) + F_r(\omega) = \frac{1}{1+j\omega} + \frac{1}{1-j\omega} = \frac{2}{1+\omega^2}$$

Part b. Find the Continuous-Time Fourier Transform of $f_2(t)$:

$$f_2(t) = \begin{cases} te^{-t} & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F_2(\omega) = \int_0^{\infty} te^{-t}e^{-j\omega t}dt = \int_0^{\infty} te^{-(1+j\omega)t}dt$$

Integrate by parts: $\int udv = uv - \int vdu$ where

$$u = t$$
$$dv = e^{-(1+j\omega)t}dt$$

so that

$$du = dt$$
$$v = \frac{e^{-(1+j\omega)t}}{-(1+j\omega)}$$

$$\begin{aligned} F_2(\omega) &= -t\frac{e^{-(1+j\omega)t}}{1+j\omega}\Bigg|_0^{\infty} - \int_0^{\infty} \frac{e^{-(1+j\omega)t}}{-(1+j\omega)}dt \\ &= \int_0^{\infty} \frac{e^{-(1+j\omega)t}}{1+j\omega}dt = -\frac{e^{-(1+j\omega)t}}{(1+j\omega)^2}\Bigg|_0^{\infty} = \frac{1}{(1+j\omega)^2} \end{aligned}$$

Part c. Find the Discrete-Time Fourier Transform of $f_3[n]$:

$$f_3[n] = \left(\frac{1}{2}\right)^{|n|}$$

$$
\begin{aligned}
F_3(\Omega) &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|n|} e^{-j\Omega n} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} e^{-j\Omega n} + \sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^{-n} e^{-j\Omega n} - 1 \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} e^{-j\Omega n} + \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{m} e^{j\Omega m} - 1 \\
&= \frac{1}{1-\frac{1}{2}e^{-j\Omega}} + \frac{1}{1-\frac{1}{2}e^{j\Omega}} - 1 \\
&= \frac{(1-\frac{1}{2}e^{j\Omega}) + (1-\frac{1}{2}e^{-j\Omega}) - (1-\frac{1}{2}e^{j\Omega})(1-\frac{1}{2}e^{-j\Omega})}{(1-\frac{1}{2}e^{j\Omega})(1-\frac{1}{2}e^{-j\Omega})} \\
&= \frac{2-\frac{1}{2}e^{j\Omega}-\frac{1}{2}e^{-j\Omega}-1+\frac{1}{2}e^{j\Omega}+\frac{1}{2}e^{-j\Omega}-\frac{1}{4}}{(1-\frac{1}{2}e^{j\Omega})(1-\frac{1}{2}e^{-j\Omega})} \\
&= \frac{\frac{3}{4}}{1-\frac{1}{2}e^{-j\Omega}-\frac{1}{2}e^{-j\Omega}+\frac{1}{4}} \\
&= \frac{\frac{3}{4}}{\frac{5}{4}-\cos(\Omega)} = \frac{3}{5-4\cos(\Omega)}
\end{aligned}
$$

Part d. Find the Discrete-Time Fourier Transform of $f_4[n]$:

$$
f_4[n] = \begin{cases} n\left(\frac{1}{2}\right)^n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}
$$

$$
\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}
$$

Differentiate both sides by a:

$$
\sum_{n=0}^{\infty} n a^{n-1} = \frac{1}{(1-a)^2}
$$

It follows that

$$
\sum_{n=0}^{\infty} n a^{n} = \frac{a}{(1-a)^2}
$$

$$
F_4(\Omega) = \sum_{n=0}^{\infty} n \left(\frac{1}{2}\right)^n e^{-j\Omega n} = \sum_{n=0}^{\infty} n \left(\frac{1}{2}e^{-j\Omega}\right)^n = \frac{\frac{1}{2}e^{-j\Omega}}{\left(1-\frac{1}{2}e^{-j\Omega}\right)^2}
$$

Part e. Find the Fourier transform of $f_3(t)$:

$$
f_3(t) = \frac{1}{1+t^2}
$$

$$
\pi e^{-|\omega|}
$$