Discrete Fourier Transforms

Using an analysis width $N = 32$, determine the DFTs of each of the following signals.

$$x_1[n] = \cos(\frac{\pi}{4}n)$$

$$X_1[k] = \frac{1}{32}\sum_{n=0}^{31}\cos\left(\frac{\pi}{4}n\right)e^{-j2\pi kn/32} = \frac{1}{32}\sum_{n=0}^{31}\frac{1}{2}\left(e^{j\pi n/4}+e^{-j\pi n/4}\right)e^{-j2\pi kn/32}$$
$$= \frac{1}{64}\sum_{n=0}^{31}e^{-j2\pi(k-4)n/32} + \frac{1}{64}\sum_{n=0}^{31}e^{-j2\pi(k+4)n/32}$$
$$= \frac{1}{64}\left(\frac{1-e^{-j2\pi(k-4)}}{1-e^{-j2\pi(k-4)/32}}\right) + \frac{1}{64}\left(\frac{1-e^{-j2\pi(k+4)}}{1-e^{-j2\pi(k+4)/32}}\right)$$

The numerator of the first term above is zero for all values of k because $2\pi(k-4)$ is an integer multiple of 2π for all integer values of k. The denominator of the first term is zero when $k-4$ is an integer multiple of 32. Therefore when $k = 4$, the first term is 0/0, which we can evaluate using L'Hôpital's rule. Similar logic applies to the second term above when $k = 28$. The final answer is

$$X_1[k] = \frac{1}{2}\delta[k-4] + \frac{1}{2}\delta[k-28]$$

An alternative approach to solving this problem is to realize that the signal $x_1[n] = \cos(\pi n/4)$ is a fourth harmonic of the fundamental frequency $2\pi/32$. Therefore we can write $X_1[k]$ as $A\delta[k-4] + B\delta[k-28]$ where A and B are constants. Substituting this expression into the synthesis equation yields

$$x_1[n] = \sum_{k=0}^{31}X_1[k]e^{j2\pi kn/32} = \sum_{k=0}^{31}\Big(A\delta[k-4]+B\delta[k-28]\Big)e^{j2\pi kn/32}$$
$$= Ae^{j2\pi 4n/32} + Be^{2\pi 28n/32} = Ae^{j2\pi 4n/32} + Be^{-2\pi 4n/32}$$
$$= A\Big(\cos(\pi n/4) + j\sin(\pi n/4)\Big) + B\Big(\cos(\pi n/4) + j\sin(-\pi n/4)\Big)$$
$$= (A+B)\cos(\pi n/4) + j(A-B)\sin(\pi n/4)$$

Therefore $A + B = 1$ and $A - B = 0$ yielding $A = B = 1/2$. It follows that

$$X_1[k] = \frac{1}{2}\delta[k-4] + \frac{1}{2}\delta[k-28]$$

$$x_2[n] = \sin(\frac{\pi}{4}n)$$

$$X_2[k] = \frac{1}{32}\sum_{n=0}^{31}\sin\left(\frac{\pi}{4}n\right)e^{-j2\pi kn/32} = \frac{1}{32}\sum_{n=0}^{31}\frac{1}{2j}\left(e^{j\pi n/4}-e^{-j\pi n/4}\right)e^{-j2\pi kn/32}$$
$$= \frac{1}{64j}\sum_{n=0}^{31}e^{-j2\pi(k-4)n/32} - \frac{1}{64j}\sum_{n=0}^{31}e^{-j2\pi(k+4)n/32}$$
$$= \frac{1}{64j}\left(\frac{1-e^{-j2\pi(k-4)}}{1-e^{-j2\pi(k-4)/32}}\right) - \frac{1}{64j}\left(\frac{1-e^{-j2\pi(k+4)}}{1-e^{-j2\pi(k+4)/32}}\right)$$
$$= -\frac{j}{2}\delta[k-4] + \frac{j}{2}\delta[k-28]$$

$$x_3[n] = \cos(\frac{\pi}{8}n)$$

$x_3[n]$ is a cosine at half the frequency of $x_1[n]$:

$$X_3[k] = \frac{1}{2}\delta[k-2] + \frac{1}{2}\delta[k-30]$$

$$x_4[n] = \left(\frac{1}{2}\right)^n u[n]$$

$$\begin{aligned}
X_4[k] &= \frac{1}{32}\sum_{n=0}^{31} x_4[n]e^{-j\frac{2\pi k}{32}n} \\
&= \frac{1}{32}\sum_{n=0}^{31} \left(\frac{1}{2}\right)^n e^{-j\frac{2\pi k}{32}n} \\
&= \frac{1}{32}\sum_{n=0}^{31} \left(\frac{1}{2}e^{-j\frac{2\pi k}{32}}\right)^n \\
&= \frac{1}{32}\left(\frac{1-\left(\frac{1}{2}e^{-j\frac{2\pi k}{32}}\right)^{32}}{1-\frac{1}{2}e^{-j\frac{2\pi k}{32}}}\right) \\
&= \frac{1}{32}\left(\frac{1-\left(\frac{1}{2}\right)^{32}}{1-\frac{1}{2}e^{-j\frac{2\pi k}{32}}}\right)
\end{aligned}$$