## Continuous-Time Convolution

Each of the following parts defines a new signal $y_i(t)$ in terms of signals $f_i(t)$ and $g_i(t)$. The signals $f_i(t)$ and $g_i(t)$ are defined in terms of exponential functions and the unit step function $u(t)$:

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Determine a closed-form solution for each $y_i(t)$. Your solutions should not include integrals, convolutions, or infinite sums, but may include separate expressions for different regions of $t$.

**Part a.** Determine a closed-form expression for $y_1(t) = (f_1 * g_1)(t)$ where $f_1(t) = g_1(t) = u(t)$.

$$y_1(t) = \int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau = \int_0^t d\tau = tu(t)$$

$$y_1(t) = tu(t) = \begin{cases} t & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Notice that the convolution of any function $g(t)$ with $u(t)$ is equal to the indefinite integral of $g(t)$:

$$(g * u)(t) = \int_{-\infty}^{\infty} g(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} g(\tau)d\tau$$

Thus $(u * u)(t) = tu(t)$.

**Part b.** Determine a closed-form expression for $y_2(t) = (f_2 * g_2)(t)$ where $f_2(t) = u(t)$ and $g_2(t) = e^{-t}u(t)$.

$$y_2(t) = \int_{-\infty}^{\infty} u(\tau)e^{-(t-\tau)}u(t-\tau)d\tau$$

$$= e^{-t}\int_0^t e^{\tau}d\tau = e^{-t}(e^t - 1) = (1 - e^{-t})u(t)$$

$$y_2(t) = (1 - e^{-t})u(t)$$

Notice that this is the indefinite integral of $e^{-t}u(t)$.

**Part c.** Determine a closed-form expression for $y_3(t) = (f_3 * g_3)(t)$ where $f_3(t) = e^{-t}u(t)$ and $g_3(t) = e^{-2t}u(t)$.

$$y_3(t) = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-2(t-\tau)}u(t-\tau)d\tau$$

$$= e^{-2t}\int_0^t e^{\tau}d\tau = e^{-2t}(e^t - 1) = \left(e^{-t} - e^{-2t}\right)u(t)$$

$$y_3(t) = \left(e^{-t} - e^{-2t}\right)u(t)$$

**Part d.** Determine a closed-form expression for $y_4(t) = (f_4 * g_4)(t)$ where $f_4(t) = g_4(t) = e^{-t}u(t)$.

$$y_4(t) = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-(t-\tau)}u(t-\tau)d\tau$$

$$= e^{-t}\int_0^t d\tau = e^{-t}t = te^{-t}u(t)$$

$$y_4(t) = te^{-t}u(t)$$