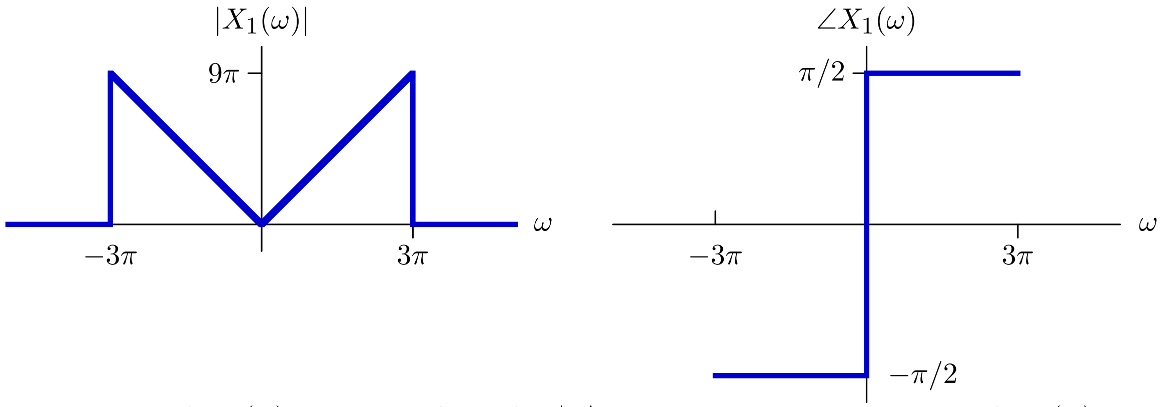
Considering Phase

Part a.

Determine $x_1(t)$, whose Fourier transform $X_1(\omega)$ has the following magnitude and angle.



Notice that the angle of $X_1(\omega)$ is only defined for $|\omega| < 3\pi$ since the magnitude of $X_1(\omega)$ is zero outside that range.

Express $x_1(t)$ as a closed-form function of time.

We can express $X_1(\omega)$ mathematically as $3j\omega$ for $|\omega| < 3\pi$ and 0 for $|\omega| > 3\pi$.

Let $X_{1a}(\omega)$ represent a rectangular pulse in frequency so that $X_{1a}(\omega) = 1$ if $|\omega| < 3\pi$ and 0 if $|\omega| > 3\pi$.

Then

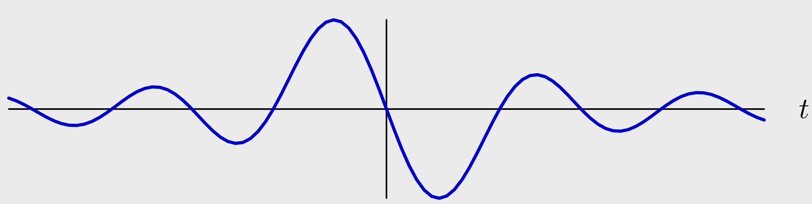
$$X_1(\omega) = 3j\omega X_{1a}(\omega)$$

Thus $x_1(t) = 3\frac{d}{dt}x_{1a}(t)$ where

$$x_{1a}(t) = \frac{\sin(3\pi t)}{\pi t}$$

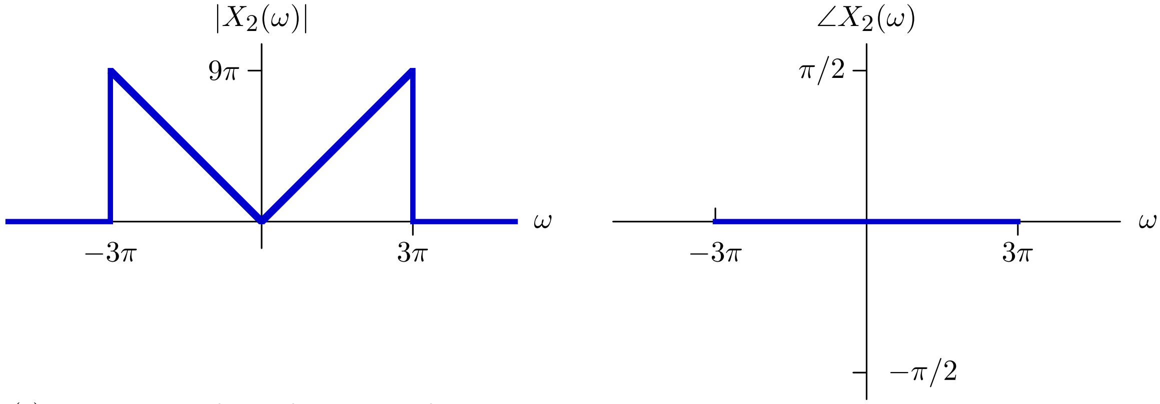
Thus

$$x_1(t) = \frac{3}{\pi t^2}\left(3\pi t \cos 3\pi t - \sin(3\pi t)\right)$$



Part b.

Determine $x_2(t)$, whose Fourier transform $X_2(\omega)$ has the following magnitude and angle.



Express $x_2(t)$ as a closed-form function of time.

Consider just the right-hand side of $X_2(\omega)$:

$$X_r(\omega) = \begin{cases} 3\omega & \text{if } 0 < \omega < 3\pi \\ 0 & \text{otherwise} \end{cases}$$

$$x_r(t) = \frac{1}{2\pi}\int_0^{3\pi} 3\omega e^{j\omega t} d\omega$$

We could integrate by parts, but it would be easier to think about $x_r(t)$ as the time derivative of some other function $y(t)$. Taking the time derivative introduces a factor of $j\omega$ in the Fourier domain, so we would like to express $x_r(t)$ in a form that has a factor of $j\omega$:

$$x_r(t) = \frac{3}{j2\pi}\int_0^{3\pi} j\omega e^{j\omega t} d\omega$$

$$y(t) = \frac{3}{j2\pi}\int_0^{3\pi} e^{j\omega t} d\omega = \frac{3}{j2\pi t}\left(1 - e^{j3\pi t}\right)$$

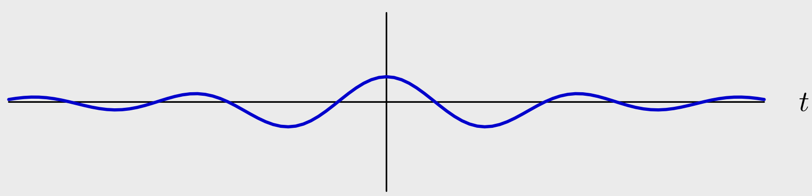
Using the product rule for derivatives, we have:

$$x_r(t) = \dot{y}(t) = \frac{3}{2\pi t}\left(-j3\pi\right) e^{j3\pi t} - \frac{3}{2\pi t^2}\left(1 - e^{j3\pi t}\right)$$

$$\begin{aligned} x(t) &= x_r(t) + x_r(-t) \\ &= -j\frac{9}{2t}e^{j3\pi t} + j\frac{9}{2t}e^{-j3\pi t} - \frac{3}{2\pi t^2}\left(1 - e^{j3\pi t}\right) - \frac{3}{2\pi t^2}\left(1 - e^{-j3\pi t}\right) \end{aligned}$$

Thus

$$x(t) = \frac{9}{t}\sin\left(3\pi t\right) - \frac{3}{\pi t^2} + \frac{3}{\pi t^2}\cos\left(3\pi t\right)$$



Part c.

What are important similarities and differences between $x_1(t)$ and $x_2(t)$? How do those similarities and differences manifest in their Fourier transforms?

Both $x_1(t)$ and $x_2(t)$ are real functions of time. However, $x_1(t)$ is an antisymmetric function of time and $x_2(t)$ is a symmetric function of time. Taken together, these features mean that $X_1(\omega)$ is an antisymmetric function of ω that is purely imaginary, and $X_2(\omega)$ is a symmetric function of ω that is purely real.

Both $X_1(\omega)$ and $X_2(\omega)$ are zero for $|\omega| > 3\pi$. Therefore, both $x_1(t)$ and $x_2(t)$ have infinite extents in time.