Convolution: Three Ways

The signal $x[n]$, defined below, is zero outside the indicated range.

Consider three ways to calculate the convolution of $x[n]$ with itself.

1. direct convolution:

$$y_1[n] = (x * x)[n] = \sum_{m=-\infty}^{\infty} x[m]x[n-m]$$

2. using DTFTs:

$$y_2[n] = \frac{1}{2\pi} \int_{2\pi} X^2(\Omega)e^{j\Omega n} d\Omega \quad \text{where} \quad X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

3. using DFTs of length $N=16$:

$$y_3[n] = 16 \sum_{k=0}^{15} X^2[k]e^{j\frac{2\pi k}{16} n} \quad \text{where} \quad X[k] = \frac{1}{16} \sum_{n=0}^{15} x[n]e^{-j\frac{2\pi k}{16} n}$$
Convolution: Three Ways

The plots on the right show the **first ten samples** of five signals. Match the signals on the left with the corresponding plots on the right.

\[
y_1 = (x \ast x)
\]

\[
y_2 = \text{DTFT}^{-1}(X^2(\Omega))
\]

\[
y_3 = N \times \text{DFT}^{-1}(X^2[k])
\]
Circular Convolution

Multiplication of DFTs corresponds to \textit{circular} convolution in time. Assume that $F[k]$ is the product of the DFTs of $f_a[n]$ and $f_b[n]$.

$$f[n] = \sum_{k=0}^{N-1} F[k] e^{j \frac{2\pi k}{N} n} = \sum_{k=0}^{N-1} F_a[k] F_b[k] e^{j \frac{2\pi k}{N} n}$$

$$= \sum_{k=0}^{N-1} F_a[k] \left( \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] e^{-j \frac{2\pi k}{N} m} \right) e^{j \frac{2\pi k}{N} n}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] \sum_{k=0}^{N-1} F_a[k] e^{j \frac{2\pi k}{N} (n-m)}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] f_{ap}[n-m]$$

where $f_{ap}[n] = f_a[n \mod N]$ is a periodically extended version of $f_a[n]$. We refer to this as \textit{circular} or \textit{periodic} convolution:

$$\frac{1}{N} (f_a \ast f_b)[n] \xrightarrow{\text{DFT}} F_a[k] \times F_b[k]$$
**Circular Convolution**

Circular convolution is equivalent to conventional convolution followed by periodic summation of results back into base period.

\[ f_a[n] \ast f_b[n] \]

\[ (f_a \bigcirc \ast f_b)[n] \]

\[ (f_a \circ \ast f_b)[n] \]
Circular Convolution

Circular convolution of two signals is equal to conventional convolution of one signal with a periodically extended version of the other.

\[ f_a[n] \ast f_b[n] = (f_a \ast f_b)[n] \]

\[ f_a[n] \ast f_b[n \mod N] = (f_a \ast f_b)[n] \]
Summary

One of the most useful properties of the DTFT is its filter property: convolution in time corresponds to multiplication in frequency.

\[(f * g)[n] \xrightarrow{\text{DTFT}} F(\Omega)G(\Omega)\]

The DFT is slightly more complicated since the DFT is equivalent to the DTFS of a periodicaly extended version of \(x[n]\):

\[x[n] = x[n + mN] \text{ for all integers } m\]

A result of this periodicity is that the convolution that results when two DFTs are multiplied is also periodic.

We refer to this type of convolution as “circular convolution.”

\[\frac{1}{N} (f \circledast g)[n] \xrightarrow{\text{DFT}} F[k]G[k]\]