6.003: Signal Processing

Superposition and Convolution

\[ y(t) = (h \ast x)(t) = \int h(\tau)x(t - \tau) \, d\tau \]

\[ y[n] = (h \ast x)[n] = \sum_{m} h[m]x[n - m] \]
Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

Start with 3 barrels of wine: newest at left, oldest at right.

\[ -1 \quad -2 \quad -3 \]
Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

Sell half of the oldest stock.

\[ -1 \quad -2 \quad -3 \quad \text{sell} \]
Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

Refill oldest barrel from next-to-oldest barrel.
Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

Refill next-to-oldest barrel from youngest barrel.
Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

Refill youngest barrel with this year’s harvest.
Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

Old and new contents mix; ready for next year.

\[
\begin{array}{c}
0 \\
-1 \\
-2 \\
-3
\end{array}
\begin{array}{c}
-1 \\
-2 \\
-3
\end{array}
\begin{array}{c}
-2 \\
-3
\end{array}
\]
Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

Old and new contents mix; ready for next year.

Properties of solera process:
• Mixing produces a more uniform product.
• Mitigates worst-case results of one bad year.
• Blends wines from MANY previous years.
We can analyze these effects with a tracer experiment. Add 1 unit of tracer to new crop; track tracer through the system.

How much tracer will be in each barrel at the end of year 3?
Add 1 unit of tracer to new crop; track tracer through the system.

<table>
<thead>
<tr>
<th>Year</th>
<th>Tracer in</th>
<th>Barrel #1</th>
<th>Barrel #2</th>
<th>Barrel #3</th>
<th>Tracer out</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>x[n]</td>
<td>y[n]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1/2</td>
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</tr>
<tr>
<td>4</td>
<td>0</td>
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<td>4/16</td>
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<td>3/16</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>1/32</td>
<td>5/32</td>
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<td>6/32</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>1/64</td>
<td>6/64</td>
<td>15/64</td>
<td>10/64</td>
</tr>
</tbody>
</table>

\[ x[n] \]

\[ y[n] \]
How would results change if tracer were added in year 1 (not 0)?

Original response:

\[ x[n] \quad y[n] \]

\[ \begin{align*}
1 \\
\ldots
\end{align*} \]

\[ n \quad n \]

Delayed input \( \rightarrow \) delayed output:

\[ x[n - 1] \quad y[n - 1] \]

\[ \begin{align*}
1 \\
\ldots
\end{align*} \]

\[ n \quad n \]

Delivering the input by a year simply delays the outputs by one year.
Time-Invariance

A system is time-invariant if delaying the input to the system simply delays the output by the same amount of time.

Given

Given

\[ x[n] \rightarrow \text{system} \rightarrow y[n] \]

the system is \textbf{time invariant} if

\[ x[n - n_0] \rightarrow \text{system} \rightarrow y[n - n_0] \]

is true for all \( n_0 \).
Solera Analysis

Scaling the input amplitudes:

\[ 0.5x[n] \]

\[ 0.5y[n] \]

Adding two inputs:

\[ x[n] + x[n-6] \]

\[ y[n] + y[n-6] \]

Linearly combining two inputs:

\[ x[n] + 0.5x[n-6] \]

\[ y[n] + 0.5y[n-6] \]
Linearity

A system is linear if its response to a weighted sum of inputs is equal to the weighted sum of its responses to each of the inputs.

Given

\[ x_1[n] \rightarrow \text{system} \rightarrow y_1[n] \]

and

\[ x_2[n] \rightarrow \text{system} \rightarrow y_2[n] \]

the system is linear if

\[ \alpha x_1[n] + \beta x_2[n] \rightarrow \text{system} \rightarrow \alpha y_1[n] + \beta y_2[n] \]

is true for all \( \alpha \) and \( \beta \).
If a system is linear and time invariant, the response to an arbitrary input is the convolution of that input with the unit-sample response.

\[
\delta[n] \rightarrow \text{system} \rightarrow h[n]
\]

\[
\delta[n-k] \rightarrow \text{system} \rightarrow h[n-k]
\]

\[
x[k] \delta[n-k] \rightarrow \text{system} \rightarrow x[k] h[n-k]
\]

\[
x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \rightarrow \text{system} \rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]
\]
## Check Yourself

For solera process ...

<table>
<thead>
<tr>
<th>Year $n$</th>
<th>Tracer in $x[n]$</th>
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<th>Barrel #2</th>
<th>Barrel #3</th>
<th>Tracer out $y[n]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
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</table>

**How much tracer would be found in output from year 5, if one unit of tracer is added in years 0, 1, and 2?**

1. $\frac{21}{32}$
2. $\frac{1}{2}$
3. $\frac{3}{16}$
4. $\frac{9}{16}$
5. none of above
Check Yourself

For solera process ...

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<th>Tracer out $y[n]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
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</tr>
<tr>
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How much tracer would be found in output from year 5, if one unit of tracer is added in years 0, 1, and 2? 2

1. $\frac{21}{32}$  
2. $\frac{1}{2}$  
3. $\frac{3}{16}$  
4. $\frac{9}{16}$  
5. none of above
Divide and Conquer

The content of barrel #3 has no direct dependence on barrel #1. The new content of barrel #3 depends only on itself and barrel #2. All dependence on barrel #1 is through barrel #2.

Since barrel #3 depends only on barrel #2, and barrel #2 depends only on barrel #1, the three barrel system is equivalent to the cascade of three one-barrel systems!
Divide and Conquer

Making a three-barrel system by cascading three one-barrel systems.

$$x[n] \rightarrow \text{3-barrel solera} \rightarrow y[n]$$

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<tr>
<td>$n$</td>
<td>$x[n] = \delta[n]$</td>
<td>#1</td>
<td>#2</td>
<td>#3</td>
<td>$y[n] = h_3[n]$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
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$$x[n] = \delta[n] \quad y[n] = h_3[n]$$
Divide and Conquer

Find the unit-sample response $h_1[n]$ for a 1-barrel solera.

Show that $h_3[n] = ((h_1 * h_1) * h_1)[n]$.
Divide and Conquer

Find the unit-sample response $h_1[n]$ for a 1-barrel solera.

$$x[n] \rightarrow \text{1-barrel solera} \rightarrow \text{1-barrel solera} \rightarrow \text{1-barrel solera} \rightarrow y[n]$$

Show that $h_3[n] = ((h_1 \ast h_1) \ast h_1)[n]$.

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n-1] = \begin{cases} \left(\frac{1}{2}\right)^n & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases}$$

No tracer leaves barrel \#1 on the year the tracer is added ($n = 0$).
Half leaves the following year.
Half of the remainder leaves on each subsequent year.
The sum of all that leaves (from $n = 0$ to $\infty$) is 1 (all of it).
Divide and Conquer

\[ h_1[n] = \left(\frac{1}{2}\right)^n u[n - 1] \]

\[ h_2[n] = (h_1 \ast h_1)[n] = \sum_{m=-\infty}^{\infty} h_1[m] h_1[n - m] \]

\[ = \sum_{m=-\infty}^{\infty} \left(\frac{1}{2}\right)^m u[m-1] \left(\frac{1}{2}\right)^{n-m} u[n-m-1] \]

The value being summed is zero unless \( m - 1 \geq 0 \) and \( n - m - 1 \geq 0 \). Therefore \( 1 \leq m \leq n - 1 \) and \( n \geq 2 \):

\[ h_2[n] = \sum_{m=1}^{n-1} \left(\frac{1}{2}\right)^n = (n-1) \left(\frac{1}{2}\right)^n u[n-2] \]
Divide and Conquer

\[ h_1[n] = \left(\frac{1}{2}\right)^n u[n-1] \quad \text{and} \quad h_2[n] = (n-1) \left(\frac{1}{2}\right)^n u[n-2] \]

\[ h_3[n] = (h_1 \ast h_2)[n] = \sum_{m=-\infty}^{\infty} h_1[m] h_2[n-m] \]

\[ = \sum_{m=-\infty}^{\infty} \left(\frac{1}{2}\right)^m u[m-1](n-m-1) \left(\frac{1}{2}\right)^{n-m} u[n-m-2] \]

The value being summed is zero unless \( m - 1 \geq 0 \) and \( n - m - 2 \geq 0 \). Therefore \( 1 \leq m \leq n - 2 \) and \( n \geq 3 \):

\[ h_3[n] = \sum_{m=1}^{n-2} (n-m-1) \left(\frac{1}{2}\right)^n = \frac{(n-1)(n-2)}{2} \left(\frac{1}{2}\right)^n u[n-3] \]

\[ h_3[n] \]

\[ \frac{3}{16} \]

\[ n \]