Please WAIT until we tell you to begin.

This exam is closed book, but you may use three 8.5 \times 11 sheets of notes (six sides).

You may NOT use any electronic devices (such as calculators and phones).

If you have questions, please come to us at the front of the room to ask.

Please enter all solutions in the boxes provided.
Work on other pages with QR codes will be considered for partial credit.
Please provide a note if you continue work on worksheets at the end of the exam.

Please do not write on the QR codes at the bottom of each page.
We use those codes to identify which pages belong to each student.

**Trigonometric Identities Reference**

\[
\begin{align*}
\cos(a + b) &= \cos(a) \cos(b) - \sin(a) \sin(b) & \cos(a - b) &= \cos(a) \cos(b) + \sin(a) \sin(b) \\
\sin(a + b) &= \sin(a) \cos(b) + \cos(a) \sin(b) & \sin(a - b) &= \sin(a) \cos(b) - \cos(a) \sin(b) \\
\cos(a) + \cos(b) &= 2\cos\left(\frac{a+b}{2}\right) \cos\left(\frac{a-b}{2}\right) & \cos(a) - \cos(b) &= -2\sin\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right) \\
\sin(a) + \sin(b) &= 2\sin\left(\frac{a+b}{2}\right) \cos\left(\frac{a-b}{2}\right) & \sin(a) - \sin(b) &= 2\cos\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right) \\
\cos(a+b) + \cos(a-b) &= 2 \cos(a) \cos(b) & \cos(a+b) - \cos(a-b) &= -2 \sin(a) \sin(b) \\
\sin(a+b) + \sin(a-b) &= 2 \sin(a) \cos(b) & \sin(a+b) - \sin(a-b) &= 2 \cos(a) \sin(b) \\
2\cos(a) \cos(b) &= \cos(a-b) + \cos(a+b) & 2\sin(a) \sin(b) &= \cos(a-b) - \cos(a+b) \\
2\sin(a) \cos(b) &= \sin(a+b) + \sin(a-b) & 2\cos(a) \sin(b) &= \sin(a+b) - \sin(a-b)
\end{align*}
\]
1 Fourier Varieties (20 points)

Part 1. The following signal

\[ x_1[n] = \sin(1.5\pi n) + 3\cos(2.3\pi n) \]

is periodic in discrete-time \( n \). Therefore it can be represented as a Fourier series of the form

\[ x_1[n] = \sum_{k=\langle N \rangle} X_1[k]e^{j\frac{2\pi k}{N}n} \]  

(1)

Determine the fundamental period \( N \) of \( x_1[n] \) and enter its value in the box below.

\[ N = \boxed{20} \]

Determine values of \( X_1[k] \) to represent \( x_1[n] \) using Equation 1. List the non-zero coefficients \( X_1[k] \) in the table below. If more than 10 coefficients are needed, list any 10 of them. If fewer than 10 are needed, leave remaining boxes empty.

<table>
<thead>
<tr>
<th>( k )</th>
<th>non-zero ( X_1[k] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-5 ) or ( 15 )</td>
<td>(-0.5j)</td>
</tr>
<tr>
<td>(-3 ) or ( 17 )</td>
<td>(1.5)</td>
</tr>
<tr>
<td>(3)</td>
<td>(1.5)</td>
</tr>
<tr>
<td>(5)</td>
<td>(0.5j)</td>
</tr>
</tbody>
</table>

The \( \sin(1.5\pi n) \) term will repeat when \( 1.5\pi n \) is an integer multiple of \( 2\pi \), which will happen for integer multiples of \( n = 4 \). The \( \cos(2.3\pi n) \) term will repeat when \( 2.3\pi n \) is an integer multiple of \( 2\pi \), which will happen for integer multiples of \( n = 20 \). Thus both terms are periodic in \( n = 20 \), and the fundamental period is \( N = 20 \).

The signal \( x_1[n] \) can be written as

\[ x_1[n] = \frac{e^{j1.5\pi n} - e^{-j1.5\pi n}}{2j} + 3 \frac{e^{j2.3\pi n} + e^{-j2.3\pi n}}{2} \]

\[ = -\frac{1}{2}je^{j2\pi 15n/20} + \frac{1}{2}je^{-j2\pi 15n/20} + \frac{3}{2}e^{j2\pi 23n/20} + \frac{3}{2}e^{-j2\pi 23n/20} \]

\[ = -\frac{1}{2}je^{-j2\pi 5n/20} + \frac{1}{2}je^{j2\pi 5n/20} + \frac{3}{2}e^{j2\pi 3n/20} + \frac{3}{2}e^{-j2\pi 3n/20} \]
Part 2. The following signal
\[ x_2(t) = \sin(1.5\pi t) + 3\cos(2.3\pi t) \]
is periodic in continuous-time \( t \). Therefore it can be represented as a Fourier series of the form
\[ x_2(t) = \sum_{k=-\infty}^{\infty} X_2[k]e^{j2\pi kt} \tag{2} \]
\[ T = 20 \]

Determine values of \( X_2[k] \) to represent \( x_2(t) \) using Equation 2. List the non-zero coefficients \( X_2[k] \) in the table below. If more than 10 coefficients are needed, list any 10 of them. If fewer than 10 are needed, leave remaining boxes empty.

<table>
<thead>
<tr>
<th>( k )</th>
<th>non-zero ( X_1[k] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-23</td>
<td>1.5</td>
</tr>
<tr>
<td>-15</td>
<td>0.5j</td>
</tr>
<tr>
<td>15</td>
<td>-0.5j</td>
</tr>
<tr>
<td>23</td>
<td>1.5</td>
</tr>
</tbody>
</table>

The \( \sin(1.5\pi t) \) term will repeat when \( 1.5\pi t \) is an integer multiple of \( 2\pi \), which will happen for integer multiples of \( t = 4 \). The \( \cos(2.3\pi t) \) term will repeat when \( 2.3\pi t \) is an integer multiple of \( 2\pi \), which will happen for integer multiples of \( t = 20 \). Thus both terms are periodic in \( n = 20 \), and the fundamental period is \( T = 20 \).

The signal \( x_2(t) \) can be written as
\[ x_2(t) = \frac{e^{j1.5\pi t} - e^{-j1.5\pi t}}{2j} + 3\frac{e^{j2.3\pi t} + e^{-j2.3\pi t}}{2} \]
\[ = -\frac{1}{2}e^{j2\pi 15t/20} + \frac{1}{2}e^{-j2\pi 15t/20} + \frac{3}{2}e^{j2\pi 23t/20} + \frac{3}{2}e^{-j2\pi 23t/20} \]
Part 3. Determine the Discrete Fourier Transform (DFT) of the following signal

\[ x_3[n] = \sin(1.5\pi n) + 3\cos(2.3\pi n) \]

using an analysis window \( N = 100 \). Determine the non-zero coefficients \( X_3[k] \) and enter their values in the table below. If there are more than 10 non-zero values, include any 10 of them. If fewer than 10 are needed, leave remaining boxes empty.

<table>
<thead>
<tr>
<th>( k )</th>
<th>non-zero ( X_1[k] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-25 or 75</td>
<td>-0.5j</td>
</tr>
<tr>
<td>-15 or 85</td>
<td>1.5</td>
</tr>
<tr>
<td>15</td>
<td>1.5</td>
</tr>
<tr>
<td>25</td>
<td>0.5j</td>
</tr>
</tbody>
</table>

The signal \( x_3[n] \) can be written as

\[
x_3[n] = \frac{e^{j1.5\pi n} - e^{-j1.5\pi n}}{2j} + 3\frac{e^{j2.3\pi n} + e^{-j2.3\pi n}}{2}
\]

\[
= -\frac{1}{2}e^{j2\pi 75n/100} + \frac{1}{2}e^{-j2\pi 75n/100} + \frac{3}{2}e^{j2\pi 115n/100} + \frac{3}{2}e^{-j2\pi 115n/100}
\]

\[
= -\frac{1}{2}e^{-j2\pi 25n/100} + \frac{1}{2}e^{j2\pi 25n/100} + \frac{3}{2}e^{j2\pi 15n/100} + \frac{3}{2}e^{-j2\pi 15n/100}
\]
Part 4. Make a plot of the Discrete-Time Fourier Transform (DTFT) of the following signal

\[ x_4[n] = \sin(1.5\pi n) + 3\cos(2.3\pi n) \]
on the axes below.
Label your plot with all of the information needed to fully specify the DTFT over the interval \([-3\pi, 3\pi]\).

Each coefficient \(X[k]\) in the Fourier series expansion of a signal corresponds to an impulse in the Fourier transform, where the weight of the impulse is \(2\pi\) times \(X[k]\) and the location of the impulse is \(\Omega = \frac{2\pi k}{N}\).

Thus the \(\sin(1.5\pi n)\) term will contribute impulses of weight \(-j\pi\) and \(j\pi\) at \(\Omega = 1.5\pi\) and \(-1.5\pi\) respectively. Similarly the \(3\cos(2.3\pi n)\) term will contribute impulses of weight \(3\pi\) at \(\Omega = \pm 2.3\pi\).

Because the DTFT is periodic in \(\Omega = 2\pi\), each of these four impulses are replicated at integer multiples of \(2\pi\). Thus there are three instances of each impulse in the interval \([-3\pi, 3\pi]\).
Part 5. Make a plot of the Continuous-Time Fourier Transform (CTFT) of the following signal

\[ x_5(t) = \sin(1.5\pi t) + 3\cos(2.3\pi t) \]
on the axes below.
Label your plot with all of the information needed to fully specify the CTFT over the interval \([-3\pi, 3\pi]\).

Each coefficient \(X[k]\) in the Fourier series expansion of a signal corresponds to an impulse in the Fourier transform, where the weight of the impulse is \(2\pi\) times \(X[k]\) and the location of the impulse is \(\omega = \frac{2\pi k}{T}\).

Thus the \(\sin(1.5\pi t)\) term will contribute impulses of weight \(-j\pi\) and \(j\pi\) at \(\omega = 1.5\pi\) and \(-1.5\pi\) respectively. Similarly the \(3\cos(2.3\pi t)\) term will contribute impulses of weight \(3\pi\) at \(\omega = \pm 2.3\pi\).
2 Slowing Down (20 points)

Let $x[n]$ represent a discrete time signal whose DTFT is given by

$$X(\Omega) = \begin{cases} 1 & \text{if } |\Omega| < \frac{\pi}{5} \\ 0 & \text{if } \frac{\pi}{5} < |\Omega| < \pi \end{cases}$$

and is periodic in $\Omega$ with period $2\pi$ as shown below.

A new signal $y_0[n]$ is derived by stretching $x[n]$ as follows:

$$y_0[n] = \begin{cases} x\left[\frac{n}{2}\right] & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

as illustrated below.

**Part 1.** Let $Y_0(\Omega)$ represent the DTFT of $y_0[n]$. Sketch the magnitude and angle of $Y_0(\Omega)$ on the axes below. Label all important parameters of your plots.

$$Y_0(\Omega) = \sum_{n=-\infty}^{\infty} y_0[n]e^{-j\Omega n} = \sum_{n \text{ even}} x\left[\frac{n}{2}\right] e^{-j\Omega n}$$

Let $n = 2m$:

$$Y_0(\Omega) = \sum_{m=-\infty}^{\infty} x[m]e^{-j\Omega 2m} = X(2\Omega)$$
**Part 2.** The $y_0[n]$ signal alternates between desired values and zeros. To reduce the effect of these zeros, we can convolve $y_0[n]$ with $h_1[n]$ given by
\[
h_1[n] = \delta[n] + \delta[n - 1]
\]
to generate a new signal
\[
y_1[n] = (y_0 * h_1)[n].
\]
Sketch $y_1[n]$ for $-15 \leq n \leq 15$ on the axes below.

Briefly describe the relation between $y_0[n]$ and $y_1[n]$.

**Relation:** The zero-values that were inserted into $y_0[n]$ have been converted to copies of the previous non-zero values – i.e., $y_1[n]$ is a zero-order hold version of $y_0[n]$.

Let $H_1(\Omega)$ represent the DTFT of $h_1[n]$. Sketch the magnitude and angle of $H_1(\Omega)$ on the axes below. Label all important parameters of your plots.

The effect of convolving $y_0[n]$ with $h_1[n]$ is to filter $Y_0(\Omega)$ by $H_1(\Omega)$. Briefly describe the effect of this filtering on $Y_0(\Omega)$.

**Effect:** When $\Omega$ is near $\pi$, the magnitude of $H_1(\Omega)$ is small. Thus $H_1(\Omega)$ reduces the high frequency components that were introduced by inserting zeros in $x[n]$. The filter also introduces a phase delay in $Y_1(\Omega)$ relative to $Y_0(\Omega)$.

\[
H_1(\Omega) = \sum_{n=-\infty}^{\infty} h_1[n]e^{-j\Omega n} = 1 + e^{-j\Omega} = e^{-j\Omega/2} \left( e^{j\Omega/2} + e^{-j\Omega/2} \right) = 2e^{-j\Omega/2} \cos \frac{\Omega}{2}
\]
Part 3. An even better way to smooth \( y_0[n] \) is to convolve it with \( h_2[n] \) given by
\[
h_2[n] = \frac{1}{2} \delta[n + 1] + \delta[n] + \frac{1}{2} \delta[n - 1]
\]
to generate a new signal
\[
y_2[n] = (y_0 * h_2)[n].
\]
Sketch \( y_2[n] \) for \(-15 \leq n \leq 15\) on the axes below.

Briefly describe the relation between \( y_0[n] \) and \( y_2[n] \).

Relation: The zero-values that were inserted into \( y_0[n] \) have been replaced with the average of the samples on either side of them, as in piecewise linear interpolation.

Let \( H_2(\Omega) \) represent the DTFT of \( h_2[n] \). Sketch the magnitude and angle of \( H_2(\Omega) \) on the axes below. Label all important parameters of your plots.

The effect of convolving \( y_0[n] \) with \( h_2[n] \) is to filter \( Y_0(\Omega) \) by \( H_2(\Omega) \). Briefly describe differences between the filtering provided by \( H_1(\cdot) \) and \( H_2(\cdot) \).

Effect: When \( \Omega \) is near \( \pi \), the magnitude of \( H_2(\Omega) \) is small. Thus \( H_2(\Omega) \) reduces the high frequency components that were introduced by inserting zeros in \( x[n] \).
However, \( H_2(\Omega) \) introduces no phase shift, while \( H_1(\Omega) \) introduced a delay. Also, the amplitudes of frequency components near \( \pi \) are reduced more by \( H_2 \) than by \( H_1 \).

\[
H_2(\Omega) = \sum_{n=-\infty}^{\infty} h_2[n]e^{-j\Omega n} = 1 + \frac{1}{2} e^{-j\Omega} + \frac{1}{2} e^{j\Omega} = 1 + \cos \Omega
\]
3 Smiley :-) (18 points)

Let $x_0[r, c]$ represent the $15 \times 15$ pixel smiley image shown in panel A below, where $r$ and $c$ represent the row and column numbers, and the point $r = c = 0$ is in the center of the image.

Each of images B-P was derived from image A, as described on the following page.

Pixel values for each image are scaled so that black and white represent the most negative and positive values for that image.
Each of the following parts describes a 2D DFT that was generated from the 2D DFT $X_0[k_r, k_c]$, of image A. For each of the following parts, enter the letter (A-P) of the image that corresponds to the 2D DFT described to the right of the box.

<table>
<thead>
<tr>
<th>Part</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>$X_1[k_r, k_c] = \text{Re}(X_0[k_r, k_c])$</td>
</tr>
<tr>
<td>O</td>
<td>$X_2[k_r, k_c] = j \text{Im}(X_0[k_r, k_c])$</td>
</tr>
<tr>
<td>D</td>
<td>$X_3[k_r, k_c] = X_0[-k_r, -k_c]$</td>
</tr>
<tr>
<td>B</td>
<td>$X_4[k_r, k_c] = X_0[k_r, k_c] \times e^{-j2\pi k_r/15}$</td>
</tr>
<tr>
<td>J</td>
<td>$X_5[k_r, k_c] = X_0[k_r, k_c] \times \cos(2\pi k_c/15)$</td>
</tr>
<tr>
<td>L</td>
<td>$X_6[k_r, k_c] = X_0[k_r, k_c] \times j \sin(2\pi k_c/15)$</td>
</tr>
<tr>
<td>M</td>
<td>$X_7[k_r, k_c] = X_0[k_r, k_c] \times \left(1 - (-1)^{k_r}\right)/2$</td>
</tr>
<tr>
<td>E</td>
<td>$X_8[k_r, k_c] = X_0[-k_c, k_r]$</td>
</tr>
<tr>
<td>F</td>
<td>$X_9[k_r, k_c] = X_0[-k_c, k_r] \times e^{j2\pi k_r/15}$</td>
</tr>
</tbody>
</table>

**X1:** Because $x_0$ is real-valued, the real part of $X_0$ corresponds to the symmetric part of $x_0$. The symmetric part of $x_0[r, c]$ is $(x_0[r, c] + x_0[-r, -c])/2 \rightarrow P$.

**X2:** Because $x_0$ is real-valued, j times the imaginary part of $X_0$ corresponds to the antisymmetric part of $x_0$. The symmetric part of $x_0[r, c]$ is $(x_0[r, c] - x_0[-r, -c])/2 \rightarrow O$.

**X3:** Negating r and c flips $x[r, c]$ vertically and horizontally $\rightarrow D$.

**X4:** Multiplying by $e^{-j2\pi k_r/15}$ shifts the image one pixel in the r direction $\rightarrow B$.

**X5:** Multiplying by $\cos(2\pi k_c/15) = \frac{1}{2} e^{j2\pi k_c/15} + \frac{1}{2} e^{-j2\pi k_c/15}$ sums the image shifted one pixel to the left with one pixel to the right $\rightarrow J$.

**X6:** Multiplying by $j \sin(2\pi k_c/15) = \frac{1}{2} e^{j2\pi k_c/15} - \frac{1}{2} e^{-j2\pi k_c/15}$ subtracts half the image shifted one pixel to the right from half the image shifted one pixel to the left $\rightarrow L$.

**X7:** Multiplying by $\left(1 - (-1)^{k_r}\right)/2$ is the same as multiplying by $\left(1 - e^{j\pi k_r}\right)$, which is equivalent to subtracting half the image shifted by half the frame in the r direction from half the original $\rightarrow M$.

**X8:** Changing the indices from $k_r, k_c$ to $-k_c, k_r$ is equivalent to rotating the DFT by a quarter turn in the clockwise direction. This rotates $x_0[r, c]$ by a quarter turn in the clockwise direction $\rightarrow E$.

**X9:** Changing the indices as in $X_8$ and then multiplying by $e^{j2\pi k_r/15}$ rotates the image by a quarter turn in the clockwise direction and then shifts the result one pixel up $\rightarrow F$. 

11
4 Bricks (22 points)

Let $b[r, c]$ represent the following $32 \times 32$ pixel image of a brick pattern, where $r$ and $c$ represent the row and column numbers of pixels in the image, and $r = c = 0$ is in the center of the image.

**Part 1.** This brick pattern $b[r, c]$ can be written as a sum of 3 images that each contain only horizontal or vertical pieces. Identify those three images from the images shown on the following page.

$$b[r, c] = A1 + A5 + B2$$

Horizontal lines (A1) + lines between bricks in the top and third rows (A5) + lines between bricks in the second and fourth rows (B2)

**Part 2.** Each of the pieces in your answer to Part 1 can be represented by its 2D DFT. Find the magnitude of the 2D DFT for each piece from the images shown on the following page, and enter their labels in the boxes below. Order your answers in this part (from left to right) to match the order of your answers in the previous part.

The DFT of horizontal lines (A1) is a column of dots at $k_c = 0$. The period in the space domain (8) corresponds to $k$ in the frequency domain. Thus there are 8 dots in the column $\rightarrow C5$.

We can think about the dashed pattern in A5 as the convolution of one vertical strip with a $2 \times 2$ array of dots – with each dot separated from its neighbors by 16 pixels in both $r$ and $c$. The DFT of this dot pattern is a $16 \times 16$ array of dots. Convolution of the vertical stripe in space is equivalent to multiplying by the DFT of the vertical stripe in frequency. Multiplication produces a $\text{sine}(k)/k$ weighting along the vertical direction $\rightarrow D4$.

B2 is just a shifted version of A5. The shift changes the phase but has no effect on magnitude. Therefore the DFT of B2 is the same as that of A5 $\rightarrow D4$. 

**Part 3.** The brick pattern $b[r, c]$ can also be constructed by circularly convolving two of the images on the following page. Identify those two images.

\[ b[r, c] = \ast \]

The brick pattern can be produced by the convolution of one brick with a dot pattern with one dot at the center of each brick. The brick pattern is $G_1$ and the centers of the bricks are shown in $G_3$.

**Part 4.** Each of the pieces in your answer to Part 3 can be represented by its 2D DFT. Find the magnitude of the 2D DFT for each piece from the images shown on the following page, and enter their labels in the boxes below. Order your answers in this part (from left to right) to match the order of your answers in the previous part.

We can think of $G_1$ as the sum of two horizontal line segments with two vertical line segments. The horizontal line segments are displaced 8 pixels vertically from each other, so there should be 8 vertical periods in the DFT. The vertical line segments are displaced 16 pixels horizontally from each other, so there should be 16 horizontal periods in the DFT. $\rightarrow E_3$.

The brick pattern $G_3$ consists of diagonally displaced dots, so we should expect that the DFT will also contain diagonally displaced dots. There are slightly more than 3 periods of dots on the diagonal in $G_3$, so the dots in the DFT should be spaced about 3 pixels apart (measured on the diagonal) $\rightarrow F_2$.

**Part 5.** Identify the image on the following page that shows the magnitude of the 2D DFT of $b[r, c]$.

The convolution of $G_1$ and $G_2$ corresponds to the multiplication of $E_3$ with $F_2 \rightarrow F_4$. 
Black represents 0 and white represents the largest value in that image.
5 Signal Changes (20 points)

Consider the following signal $x[n]$, of which only a small portion is shown, but for which $x[n] = x[n + 6]$ for all $n$:

In this problem, we will explore alternative signals that we can make by changing a small number of samples per period in the signal above. Consider the related signals below (where $x_1[\cdot]$ through $x_5[\cdot]$ represent time-domain signals and $X_1[\cdot]$ through $X_5[\cdot]$ represent the associated Fourier series).

For each, we want to know whether it is possible to create a signal with the given properties by modifying at most two samples per period of $x[n]$. If it is possible, specify which value(s) you wish to change, and what you wish to change them to. If it is not possible, put an X in each box.

There may be multiple solutions to some of the following parts. You need only find one solution for full credit.

Part 1. $x_1$

We would like to create $x_1[\cdot]$ by modifying at most two samples in $x[\cdot]$, where $X_1[\cdot]$ is a symmetric function of $k$.

<table>
<thead>
<tr>
<th>Which index should be changed?</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>What should it be changed to?</td>
<td>$-3$</td>
</tr>
<tr>
<td>Which index should be changed?</td>
<td>X</td>
</tr>
<tr>
<td>What should it be changed to?</td>
<td>X</td>
</tr>
</tbody>
</table>

If $X[k]$ is the DFT of $x[n]$, then $X[-k]$ will be the DFT of $x[-n]$. It follows that the antisymmetric part of $X[k]$ (which is $(X[k] - X[-k])/2$) is the DFT of the antisymmetric part of $x[n]$ (which is $(x[n] - x[-n])/2$). If the former is 0 for all $k$, then the latter will be zero for all $n$. Thus if we make $x_1[n]$ a symmetric function of $n$, $X_1[k]$ will be a symmetric function of $k$. 
Part 2. \( x_2 \) We would like to create \( x_2[\cdot] \) by modifying \textbf{at most two} samples in \( x[\cdot] \), where \( e^{j\frac{5\pi k}{3}}X_2[\cdot] \) is purely imaginary.

<table>
<thead>
<tr>
<th>Which index should be changed?</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>What should it be changed to?</td>
<td>3</td>
</tr>
<tr>
<td>Which index should be changed?</td>
<td>1</td>
</tr>
<tr>
<td>What should it be changed to?</td>
<td>2</td>
</tr>
</tbody>
</table>

The DFT will be purely imaginary if the signal \( x[n] \) is real and antisymmetric in \( n \). The phase term \( e^{j\frac{5\pi}{3}} = e^{-j\frac{2\pi k}{6}} \) corresponds to a delay of one sample. Therefore \( X_2[k] \) will be purely imaginary if \( x_2[n-1] \) is antisymmetric.

Part 3. \( x_3 \)

We would like to create \( x_3[\cdot] \) by modifying \textbf{at most two} samples in \( x[\cdot] \), where \( \sum_{m=0}^{17}X_3[m] = 0 \).

<table>
<thead>
<tr>
<th>Which index should be changed?</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>What should it be changed to?</td>
<td>0</td>
</tr>
<tr>
<td>Which index should be changed?</td>
<td>X</td>
</tr>
<tr>
<td>What should it be changed to?</td>
<td>X</td>
</tr>
</tbody>
</table>

This sum is over three periods. Setting this sum to zero is the same as setting \( x[0] \) to zero.
Part 4. $x_4$
We would like to create $x_4[:]$ by modifying at most two samples in $x[: ]$, where $X_4[0] = 0$.

<table>
<thead>
<tr>
<th>Which index should be changed?</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>What should it be changed to?</td>
<td>5</td>
</tr>
<tr>
<td>Which index should be changed?</td>
<td>$x$</td>
</tr>
<tr>
<td>What should it be changed to?</td>
<td>$x$</td>
</tr>
</tbody>
</table>

$X[0]$ will be zero if the average value of $x[n]$ is zero.

Part 5. $x_5$
We would like to create $x_5[:]$ by modifying at most two samples in $x[: ]$, where $X_5[k] = -X_5[k+1]$ for all $k$.

<table>
<thead>
<tr>
<th>Which index should be changed?</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>What should it be changed to?</td>
<td>0</td>
</tr>
<tr>
<td>Which index should be changed?</td>
<td>4</td>
</tr>
<tr>
<td>What should it be changed to?</td>
<td>0</td>
</tr>
</tbody>
</table>

The DFT will alternate in sign if $x[n] = 0$ for $n \neq 3$. 
Worksheet (intentionally blank)
Worksheet (intentionally blank)