6.003: Signal Processing

Spring 2022

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Unit-Sample Response

The unit-sample response is a complete description of a system.

\[ \delta[n] \xrightarrow{\text{LTI}} h[n] \]

This is a bit surprising since \( \delta[n] \) is such a simple signal.

The unit-sample signal is the shortest possible non-trivial DT signal!

The response to this simple signal determines the response to any other input signal:

\[ x[n] \rightarrow y[n] = (x * h)[n] = \sum_m x[m]h[n-m] \]

Last Time: Frequency Response

The frequency response is a third way to characterize a linear time-invariant system. This characterization is based on responses to sinusoids.

\[ \cos(\Omega n) \xrightarrow{\text{LTI}} A \cos(\Omega n + \phi) \]

The idea is to characterize a system by the way \( A \) and \( \phi \) vary with \( \Omega \).

Sinusoids differ from the unit-sample signal in important ways:

- eternal (longest possible signals) versus transient (shortest possible)
- comprises a single frequency versus a sum of all possible frequencies

Representing Systems with Difference/Differential Equations

Discrete-time systems that can be described by linear difference equations with constant coefficients are linear and time-invariant.

\[ x[n] \rightarrow \text{LTI} \rightarrow y[n] \]

\[ \sum_l c_l y[n-l] = \sum_m d_m x[n-m] \]

Continuous-time systems that can be described by linear differential equations with constant coefficients are linear and time-invariant.

\[ x(t) \rightarrow \text{LTI} \rightarrow y(t) \]

\[ \sum_l \frac{d^l y(t)}{dt^l} = \sum_m \frac{d^m x(t)}{dt^m} \]

- natural and compact representations of many systems

Unit-Sample Response and Impulse Response

Discrete-time systems that are linear and time-invariant can be completely specified by their response to a unit-sample signal.

\[ x[n] \rightarrow \text{LTI} \rightarrow y[n] \]

If \( \delta[n] \rightarrow h[n] \), then \( x[n] \rightarrow y[n] = (x * h)[n] = \sum_m x[m]h[n-m] \)

Continuous-time systems that are linear and time-invariant are completely specified by their response to a unit impulse function.

\[ x(t) \rightarrow \text{LTI} \rightarrow y(t) \]

If \( \delta(t) \rightarrow h(t) \), then \( x(t) \rightarrow y(t) = (x * h)(t) = \int x(\tau)h(t-\tau)d\tau \)

- an LTI system is completely characterized by a single signal

Context: The System Abstraction

Describe a system (physical, mathematical, or computational) by the way it transforms an input signal into an output signal.

Signal in \rightarrow \text{system} \rightarrow \text{signal out}

This abstraction is particularly powerful for linear and time-invariant systems, which are both prevalent and mathematically tractable.
**Frequency Response**

Using complex exponentials to characterize the frequency response.

\[ e^{j\theta_n} \xrightarrow{\text{LTI}} A e^{j\theta_n} \]

Notice that the complex valued \( A \) can represent both amplitude and phase. We can find \( A \) using convolution.

\[ y[n] = (x * h)[n] = \sum_{m=-\infty}^{\infty} x[n-m]h[m] = \sum_{m=-\infty}^{\infty} e^{j\Omega(n-m)}h[m] \]

\[ = e^{j\theta_n} \sum_{m=-\infty}^{\infty} h[m]e^{-j\Omega m} = H(\Omega) e^{j\theta_n} \]

The response to a complex exponential is a complex exponential with the same frequency but possibly different amplitude and phase.

The map for how a system modifies the amplitude and phase of a complex exponential input is the **Fourier transform of the unit-sample response**.

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**Check Yourself**

Find the frequency response of a three-point averager:

\[ y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1]) \]

Can we think of this as a low-pass filter? Does it pass low frequencies and block high frequencies?

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**The System Abstraction**

Describe a system (physical, mathematical, or computational) by the way it transforms an input signal into an output signal.

The system abstraction applies equally well for continuous-time signals.

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**Representing Systems with Difference/Differential Equations**

*Discrete-time* systems that can be described by linear difference equations with constant coefficients are linear and time-invariant.

\[ x[n] \xrightarrow{\text{LTI}} y[n] \]

\[ \sum_l c_l y[n-l] = \sum_m d_m x[n-m] \]

*Continuous-time* systems that can be described by linear differential equations with constant coefficients are linear and time-invariant.

\[ x(t) \xrightarrow{\text{LTI}} y(t) \]

\[ \int_a^t \frac{d^m y(t)}{dt^m} \frac{dt}{m!} = \int_a^t \frac{d^m x(t)}{dt^m} \frac{dt}{m!} \]

– natural and compact representations of many systems

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**Unit-Sample Response and Impulse Response**

*Discrete-time* systems that are linear and time-invariant can be completely specified by their response to a **unit-sample signal**.

\[ x[n] \xrightarrow{\text{LTI}} y[n] \]

If \( \delta[n] \rightarrow h[n] \), then \( x[n] \rightarrow y[n] = (x * h)[n] = \sum m x[n]h[n-m] \)

*Continuous-time* systems that are linear and time-invariant are completely specified by their response to a unit **impulse** function.

\[ x(t) \xrightarrow{\text{LTI}} y(t) \]

If \( \delta(t) \rightarrow h(t) \), then \( x(t) \rightarrow y(t) = (x * h)(t) = \int x(\tau)h(t-\tau)d\tau \)

– an LTI system is completely characterized by a single signal
**Impulse Response**

The impulse response is a **complete** description of a system.

\[ \delta(t) \rightarrow \text{LTI} \rightarrow h(t) \]

This is a bit surprising since \( \delta(t) \) is zero almost everywhere. The impulse function is the **shortest** possible non-trivial CT signal!

The response to this signal determines the response to any other input.

\[ x(t) \rightarrow y(t) = (x * h)(t) = \int x(\tau)h(t-\tau)d\tau \]

**Frequency Response**

The frequency response is a third way to characterize a linear time-invariant system. This characterization is based on responses to **sinusoids**.

\[ \cos(\omega t) \rightarrow \text{LTI} \rightarrow A \cos(\omega t + \phi) \]

The idea is to characterize a system by the way \( A \) and \( \phi \) vary with \( \omega \). Sinusoids differ from the unit-sample signal in important ways:

- **eternal** (longest possible signals) versus **transient** (shortest possible)
- comprises a **single** frequency versus a **sum** of all possible frequencies

**System Abstraction**

Two **complete** representations for linear, time-invariant systems.

**Impulse Response:** responses across time for an impulse input.

\[ h(t) \]

**Frequency Response:** responses across frequencies for sinusoidal inputs.

\[ H(\omega) \]

The **frequency response** is Fourier transform of impulse response!

**Example**

Find the frequency response of a system described by the following:

\[ y(t) + \alpha \frac{dy(t)}{dt} = x(t) \]
Check Yourself
Plot the frequency response of the following system:
\[ y(t) + \alpha \frac{dy(t)}{dt} = x(t) \]
Which of the following best describes the frequency response?
- Low frequencies are attenuated.
- High frequencies are attenuated.
- All frequencies are attenuated.
- Low frequencies are delayed.
- High frequencies are delayed.
- All frequencies are delayed.

Check Yourself
Find the frequency response of a rectangular box averager:
\[ y(t) = \frac{1}{2} \int_{t-1}^{t+1} x(\tau) d\tau \]
(This CT averager is analogous to the three-point averager in DT.)

Check Yourself
Find the frequency response of a triangular averager:
\[ g(t) \]

Summary
The Fourier transform of the response of a DT LTI system is the product of the Fourier transform of the input times the system’s frequency response.
\[ X(\Omega) \xrightarrow{\text{LTI}} Y(\Omega) \]
\[ Y(\Omega) = H(\Omega)X(\Omega) \]
The frequency response \( H(\Omega) \) is the Fourier transform of the unit-sample response \( h[n] \).

The Fourier transform of the response of a CT LTI system is the product of the Fourier transform of the input times the system’s frequency response.
\[ X(\omega) \xrightarrow{\text{LTI}} Y(\omega) \]
\[ Y(\omega) = H(\omega)X(\omega) \]
The frequency response \( H(\omega) \) is the Fourier transform of the impulse response \( h(t) \).