Results of Quiz 1 have been posted.
The 6.003 Gradesheet has been posted.
Please let us know if you find errors.
Lab 6 check-in due Friday at 4pm.

March 10, 2022

Representing Systems with Difference/Differential Equations

Discrete-time systems that can be described by linear difference equations with constant coefficients are linear and time-invariant.

\[ x[n] \xrightarrow{\text{LTI}} y[n] \]

\[ \sum_l c_l y[n-l] = \sum_m d_m x[n-m] \]

Continuous-time systems that can be described by linear differential equations with constant coefficients are linear and time-invariant.

\[ x(t) \xrightarrow{\text{LTI}} y(t) \]

\[ \sum_l \frac{d^l y(t)}{dt^l} = \sum_m d_m \frac{d^m x(t)}{dt^m} \]

– natural and compact representations of many systems

Unit-Sample Response and Impulse Response

Discrete-time systems that are linear and time-invariant can be completely specified by their response to a unit-sample signal.

\[ x[n] \xrightarrow{\text{LTI}} y[n] \]

If \( \delta[n] \rightarrow h[n] \), then \( x[n] \rightarrow y[n] = (x * h)[n] = \sum_m x[m]h[n-m] \)

Continuous-time systems that are linear and time-invariant are completely specified by their response to a unit impulse function.

\[ x(t) \xrightarrow{\text{LTI}} y(t) \]

If \( \delta(t) \rightarrow h(t) \), then \( x(t) \rightarrow y(t) = (x * h)(t) = \int x(\tau)h(t-\tau)d\tau \)

– an LTI system is completely characterized by a single signal

Context: The System Abstraction

Describe a system (physical, mathematical, or computational) by the way it transforms an input signal into an output signal.

\[ \text{signal in} \rightarrow \text{system} \rightarrow \text{signal out} \]

This abstraction is particularly powerful for linear and time-invariant systems, which are both prevalent and mathematically tractable.

Three important representations for LTI systems:

- **Difference/Differential Eq:** algebraic input/output constraint
- **Convolution:** represent system by unit-sample/impulse response
- **Filter:** represent a system by its frequency response

Unit-Sample Response

The unit-sample response is a complete description of a system.

\[ \delta[n] \xrightarrow{\text{LTI}} h[n] \]

This is a bit surprising since \( \delta[n] \) is such a simple signal.
The unit-sample signal is the shortest possible non-trivial DT signal!

\[ \delta[n] \rightarrow h[n] \]

The response to this simple signal determines the response to any other input signal:

\[ x[n] \rightarrow y[n] = (x * h)[n] = \sum_m x[m]h[n-m] \]
**LTI**

\[
LTI_{ej\Omega n} = \sum_{m=-\infty}^{\infty} h[m] e^{j\Omega n} = H(\Omega) e^{j\Omega n}
\]

Notice that the complex valued \( A \) can represent both amplitude and phase. We can find \( A \) using convolution.

\[
y[n] = (x * h)[n] = \sum_{m=-\infty}^{\infty} x[n-m]h[m] = \sum_{m=-\infty}^{\infty} e^{j\Omega(n-m)}h[m]
\]

\[
y[n] = \sum_{m=-\infty}^{\infty} h[m]e^{-j\Omega n} = H(\Omega) e^{j\Omega n}
\]

The response to a complex exponential is a complex exponential with the same frequency but possibly different amplitude and phase.

The map for how a system modifies the amplitude and phase of a complex exponential input is the **Fourier transform of the unit-sample response**.

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**Frequency Response**

The frequency response is a complete description of a system.

\[
e^{j\Omega n} \rightarrow \text{LTI} \rightarrow H(\Omega) e^{j\Omega n}
\]

This is a bit surprising since \( e^{j\Omega n} \) contains a single frequency.

\[
\cos(\Omega n) \rightarrow \text{LTI} \rightarrow A\cos(\Omega n + \phi)
\]

The frequency response can be used to find response to any input signal:

\[
X(\Omega) \rightarrow Y(\Omega) = H(\Omega)X(\Omega)
\]

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**Frequency Response**

The frequency response is a complete characterization of an LTI system.

1. One can always find the response of a system to a single frequency.

\[
e^{j\Omega n} \rightarrow \text{LTI} \rightarrow H(\Omega) e^{j\Omega n}
\]

2. Scaling the input by a constant scales the output by the same constant.

\[
X(\Omega) e^{j\Omega n} \rightarrow \text{LTI} \rightarrow X(\Omega)H(\Omega) e^{j\Omega n}
\]

3. Linearity implies that the response to a sum is the sum of the responses.

\[
x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega \rightarrow \text{LTI} \rightarrow y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)H(\Omega) e^{j\Omega n} d\Omega
\]

4. The Fourier transform of the output is \( X(\Omega)H(\Omega) \).

\[
X(\Omega) \rightarrow \text{LTI} \rightarrow X(\Omega)H(\Omega)
\]

The transform of the output is \( H(\Omega) \) times the transform of the input.

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**Frequency Response**

The frequency response is an insightful description of a system.

Example:

A low-pass filter passes frequencies near 0 and rejects those near \( \pi \).

\[
H(\Omega)
\]

Very natural way to describe audio components:

- microphones
- loudspeakers
- audio equalizers
Unit-Sample Response and Frequency Response

Two complete representations for linear, time-invariant systems.

Unit-Sample Response: responses across time for a unit-sample input.

Frequency Response: responses across frequencies for sinusoidal inputs.

The frequency response is Fourier transform of unit-sample response!

Example
Find the frequency response of a system described by the following:

\[ y[n] - \alpha y[n-1] = x[n] \]

Method 1:
Find the unit-sample response and take its Fourier transform.
\[ h[n] - \alpha h[n-1] = \delta[n] \]
Solve the difference equation for \( h[n] \).
\[ h[n] = \delta[n] + \alpha h[n-1] \]
First order \( \to \) need one initial condition: \( h[-1] = 0 \)
\[ h[0] = \delta[0] + \alpha h[-1] = 1 \]
\[ h[1] = \delta[1] + \alpha h[0] = \alpha \]
\[ h[2] = \delta[2] + \alpha h[1] = \alpha^2 \]
\[ h[3] = \delta[3] + \alpha h[2] = \alpha^3 \]
\[ h[n] = \alpha^n \delta[n] \]
\[ H(\Omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-jn\Omega} = \sum_{n=0}^{\infty} \alpha^n e^{-jn\Omega} = \sum_{n=0}^{\infty} (\alpha e^{-j\Omega})^n = \frac{1}{1-\alpha e^{-j\Omega}} \]

Example
Find the frequency response of a system described by the following:

\[ y[n] - \alpha y[n-1] = x[n] \]

Method 2:
Find the response to \( e^{j\Omega n} \) directly.
\[ x[n] = e^{j\Omega n} \]
Because the system is linear and time-invariant, the output will have the same frequency as the input, but possibly different amplitude and phase.
\[ y[n] = H(\Omega) e^{j\Omega n} \]
\[ y[n-1] = H(\Omega) e^{j(n-1)\Omega} = H(\Omega) e^{-j\Omega} e^{j\Omega n} \]
Substitute into the difference equation.
\[ H(\Omega) e^{j\Omega n} - \alpha H(\Omega) e^{-j\Omega} e^{j\Omega n} = H(\Omega) (1-\alpha e^{-j\Omega}) e^{j\Omega n} = e^{j\Omega n} \]
Since \( e^{j\Omega n} \) is never 0, we can divide it out.
\[ H(\Omega) = \frac{1}{1-\alpha e^{-j\Omega}} \]
Same answer as method 1.

Check Yourself
Plot the frequency response.
\[ H(\Omega) = \frac{1}{1-\alpha e^{-j\Omega}} \]
Assume \( 0 \leq \alpha \leq 1 \).

Which of the following best describes the frequency response?

- Low frequencies are amplified.
- High frequencies are amplified.
- All frequencies are amplified.
- Low frequencies are delayed.
- High frequencies are delayed.
- All frequencies are delayed.
Summary
The Fourier transform of the response of a DT LTI system is the product of the Fourier transform of the input times the system’s \textit{frequency response}.

\[ X(\Omega) \xrightarrow{\text{LTI}} Y(\Omega) \]

\[ Y(\Omega) = H(\Omega)X(\Omega) \]

The frequency response \( H(\Omega) \) is the Fourier transform of the unit-sample response \( h[n] \).