6.003: Signal Processing

Systems

- System Abstraction
- Linearity and Time Invariance

Announcements:

- Quiz 1: March 8, 2-4pm
  - Coverage up to and including March 1 and HW4.
  - Closed book except for one page of notes (8.5"x11" both sides).
  - No electronic devices. (No headphones, cellphones, calculators, ...)
- No HW5
- A practice quiz has been posted.
  - Not turned in, not graded.
  - Solutions will be posted on Friday.
- If you have personal or medical difficulties, please contact S 3 and/or 6.003-instructors@mit.edu for accommodations.

March 1, 2022

Example: Mass and Spring

![Mass and Spring System](image)

Example: Tanks

![Tanks System](image)

Example: Cell Phone System

![Cell Phone System](image)

Signals and Systems: Widely Applicable

The Signals and Systems approach has broad application: electrical, mechanical, optical, acoustic, biological, financial, ...

![Signals and Systems](image)
Signals and Systems: Modular
The representation does not depend upon the physical substrate.

System Abstraction
The system abstraction builds on and extends our work with signals.

Properties of Systems
We will focus primarily on systems that have two important properties:
• linearity
• time invariance
Such systems are both useful and mathematically tractable.

Example: Three-Point Averaging
The output at time $n$ is average of inputs at times $n-1$, $n$, and $n+1$.

$$y[n] = \frac{1}{3} (x[n-1] + x[n] + x[n+1])$$

Think of this process as a system with input $x[n]$ and output $y[n]$.
Additivity
A system is additive if its response to a sum of signals is equal to the sum of the responses to each signal taken one at a time.

Given
\[ x_1[n] \rightarrow \text{system} \rightarrow y_1[n] \]
and
\[ x_2[n] \rightarrow \text{system} \rightarrow y_2[n] \]
the system is additive if
\[ x_1[n] + x_2[n] \rightarrow \text{system} \rightarrow y_1[n] + y_2[n] \]
is true for all possible inputs and all times \( n \).

Example: the three-point averager is additive.
The three-point average of the sum of two signals is equal to the sum of the three-point averages of the individual signals.

Homogeneity
A system is homogeneous if multiplying its input signal by a constant multiplies the output signal by the same constant.

Given
\[ x_1[n] \rightarrow \text{system} \rightarrow y_1[n] \]
the system is homogeneous if
\[ \alpha x_1[n] \rightarrow \text{system} \rightarrow \alpha y_1[n] \]
is true for all \( \alpha \) and all possible inputs and all times \( n \).

Example: the three-point averager is homogeneous.
Doubling an input signal doubles its three-point average.

Linearity
A system is linear if its response to a weighted sum of input signals is equal to the weighted sum of its responses to each of the input signals.

Given
\[ x_1[n] \rightarrow \text{system} \rightarrow y_1[n] \]
and
\[ x_2[n] \rightarrow \text{system} \rightarrow y_2[n] \]
the system is linear if
\[ \alpha x_1[n] + \beta x_2[n] \rightarrow \text{system} \rightarrow \alpha y_1[n] + \beta y_2[n] \]
is true for all \( \alpha \) and \( \beta \) and all possible inputs and all times \( n \).

A system is linear if it is both additive and homogeneous.

Time-Invariance
A system is time-invariant if delaying the input signal simply delays the output signal by the same amount of time.

Given
\[ x[n] \rightarrow \text{system} \rightarrow y[n] \]
the system is time invariant if
\[ x[n-n_0] \rightarrow \text{system} \rightarrow y[n-n_0] \]
is true for all \( n_0 \) and for all possible inputs and all times \( n \).

Example: The three-point averager is time-invariant.
Shifting the input to a 3-pt averager simply shifts the output by that same amount.

Representing Systems with Difference Equations
Consider a system represented by the following difference equation:
\[ y[n] = x[n] + x[n-1] \]
for all \( n \).
Is this system linear?

Representing Systems with Difference Equations
Determine linearity from a difference equation representation.

Example 2.
\[ y[n] = x[n] \times x[n-1] \]
for all \( n \).
Is this system linear?
Example 3: \( y[n] = nx[n] \) for all \( n \).

Is the system linear?

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\[ y[n] = nx[n] \]
for all \( n \).

Is the system \textit{linear}?

Example 3:
\[ y[n] = nx[n] \]
for all \( n \).

Is the system \textit{time-invariant}?

Assume that a system can be represented by a linear difference equation with constant coefficients.
\[
\sum_{l} c_{l} y[n - l] = \sum_{m} d_{m} x[n - m]
\]

Is such a system linear?

Is such a system time invariant?

Consider a system that is defined by
\[ y[n] = x[n] + 1 \]

Is this system linear?

Is this system time invariant?

The unit-sample signal \( \delta[n] \) is arguably the simplest non-trivial signal:
\[
\delta[n] = \begin{cases} 
1 & \text{if } n = 0 \\
0 & \text{otherwise} 
\end{cases}
\]

It is zero at all but one time point \( n = 0 \), where its value is 1.

\[
\delta[n] = \begin{cases} 
1 & \text{if } n = 0 \\
0 & \text{otherwise} 
\end{cases}
\]

\[ n \]

The unit-sample response of a system is defined as the output \( y[n] = h[n] \) that results when the input \( x[n] = \delta[n] \).
**Unit-Sample Response**

Find the unit-sample response of a three-point averager.
The output at time \( n \) is average of inputs at times \( n-1 \), \( n \), and \( n+1 \).

\[
y[n] = \frac{1}{3} (x[n-1] + x[n] + x[n+1])
\]

**Summary: System Abstraction**

The system abstraction builds on and extends our work with signals.

**Goal:** characterize a **system** to better understand the relation between two signals.

Three representations for systems:

- **Difference Equation:** algebraic constraint on samples
- **Convolution:** represent a system by its unit-sample response
- **Filter:** represent a system by its frequency response