As $T \to \infty$ find Fourier series coefficients $F_p[k]$:

$$F_p[k] = \frac{1}{T} \int_{-S}^{S} e^{-j2\pi k t} dt = \frac{1}{T} \frac{e^{-j2\pi k T} - e^{j2\pi k T}}{-j2\pi k} = \frac{2\sin(\frac{2\pi k S}{T})}{T}$$

Plot the resulting Fourier coefficients when $S=1$ and $T=8$.

Then $TF_p[k]$ represents samples of $F(\omega)$ with increasing resolution in $\omega$.

There are twice as many samples per period of the sin function. (The red samples are at new intermediate frequencies.) The amplitude is halved.
Fourier Representations of Aperiodic Signals

How do we find $F(\omega)$?

We started with a periodic version of $f(t)$ that has a FS:

$$f_p(t) \overset{\text{FT}}{\rightarrow} F_p[k]$$

Take the limit as $T \rightarrow \infty$ of both sides:

$$\lim_{T \rightarrow \infty} f_p(t) = f(t) \overset{\text{FT}}{\rightarrow} \lim_{T \rightarrow \infty} F_p[k] = \lim_{T \rightarrow \infty} \left[ \frac{1}{T} F(\omega) \right]_{\omega=\frac{2\pi}{T}}$$

Try

$$F(\omega) = \lim_{T \rightarrow \infty} TF_p[k] \Big|_{\omega=\frac{2\pi}{T}}$$

$$= \lim_{T \rightarrow \infty} T \left[ \frac{1}{T} \int_{-T/2}^{T/2} f_p(t)e^{-j2\pi k T t} \, dt \right]_{\omega=\frac{2\pi}{T}}$$

$$= \lim_{T \rightarrow \infty} \int_{-T/2}^{T/2} f_p(t)e^{-j\omega t} \, dt$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} \, dt$$

Continuous-Time Fourier Representations

Fourier series and transforms are similar: both represent signals by their frequency content.

Continuous-Time Fourier Series

$$F[k] = \frac{1}{T} \int_{-T/2}^{T/2} f(t)e^{-j2\pi k t} \, dt$$

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} F[k]e^{j2\pi k t}$$

where $\omega_0 = \frac{2\pi}{T}$

Continuous-Time Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} \, dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t} \, d\omega$$

Continuous-Time Fourier Representations

Harmonic frequencies $k\omega_0$ are samples of continuous frequency $\omega$.

Continuous-Time Fourier Series

$$F[k] = \frac{1}{T} \int_{-T/2}^{T/2} f(t)e^{-j2\pi k t} \, dt$$

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} F[k]e^{j2\pi k t}$$

where $\omega_0 = \frac{2\pi}{T}$

Continuous-Time Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} \, dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t} \, d\omega$$

Continuous-Time Fourier Representations

All of the information in a periodic signal is contained in one period. The information in an aperiodic signal is spread across all time.

Continuous-Time Fourier Series

$$F[k] = \frac{1}{T} \int_{-T/2}^{T/2} f(t)e^{-j2\pi k t} \, dt$$

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} F[k]e^{j2\pi k t}$$

where $\omega_0 = \frac{2\pi}{T}$

Continuous-Time Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} \, dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t} \, d\omega$$

Continuous-Time Fourier Representations

We can reconstruct $f(t)$ from $F(\omega)$ using a Riemann sum.

$$f(t) \overset{\text{FT}}{\rightarrow} F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} \, dt$$

$$\int_{-T/2}^{T/2} f(t)e^{-j\omega t} \, dt \Rightarrow \sum_{k=-\infty}^{\infty} F[k]e^{j2\pi k t}(\frac{2\pi}{T}) = \frac{1}{T} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t} \, d\omega$$

$$TF_p[k] = F(\omega)$$

Fourier Transform relation: $f(t) \overset{\text{FT}}{\rightarrow} F(\omega)$
Fourier transforms offer an alternative view of an aperiodic signal.

A signal and its Fourier transform contain exactly the same information, but some information is more easily seen in one domain than in the other.

There are many properties of Fourier transforms. These properties summarize systematic relations between time and frequency representations.

### Properties of Fourier Transforms

**Time delay maps to linear phase delay of the Fourier transform.**

If \( f(t) \xrightarrow{\text{FT}} F(\omega) \)
then \( f(t-\tau) \xrightarrow{\text{FT}} e^{-j\omega \tau} F(\omega) \)

\[
F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} \, dt \\
G(\omega) = \int_{-\infty}^{\infty} f(t-\tau)e^{-j\omega t} \, dt
\]

Let \( u = t-\tau \) (and therefore \( du = dt \) since \( \tau \) is a constant)

\[
G(\omega) = \int_{-\infty}^{\infty} f(u)e^{-j\omega (u+\tau)} \, du = e^{-j\omega \tau} \int_{-\infty}^{\infty} f(u)e^{-j\omega u} \, du = e^{-j\omega \tau} F(\omega)
\]

The angle of \( e^{-j\omega \tau} = -\omega \tau \).

Why does time delay change phase by an amount proportional to frequency?

### Properties of Fourier Transforms

**Doubling the frequency of a sinusoid doubles the change in phase associated with a given time delay.**

Doubling the frequency of a sinusoid doubles the change in phase associated with a given time delay.

### Properties of Fourier Transforms

**Scaling time.**

Consider the following signal and its Fourier transform.

**Time representation:**

\[
f_1(t) = \begin{cases} 1 & -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}
\]

**Frequency representation:**

\[
F_1(\omega) = \frac{2 \sin \omega}{\omega}
\]

How would these functions scale if time were stretched?

---

**Example**

Find the Fourier Transform (FT) of a rectangular pulse:

\[
f(t) = \begin{cases} 1 & -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}
\]

\[
F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} \, dt = \int_{-1}^{1} e^{-j\omega t} \, dt = \frac{2 \sin \omega}{\omega}
\]

\[
F(\omega) \text{ provides a recipe for constructing } f(t) \text{ from sinusoidal components:}
\]

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t} \, d\omega
\]

A square pulse contains (almost) all frequencies \( \omega \) (missing just \( \pi, 2\pi, \ldots \)).

---

**Check Yourself**

Signal \( f_2(t) \) and its Fourier transform \( F_2(\omega) \) are shown below.

<table>
<thead>
<tr>
<th>( f_2(t) )</th>
<th>( F_2(\omega) )</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="signal.png" alt="Signal" /></td>
<td><img src="fourier_transform.png" alt="Fourier Transform" /></td>
</tr>
</tbody>
</table>

Which of the following is true?

1. \( b = 2 \) and \( \omega_0 = \pi/2 \)
2. \( b = 2 \) and \( \omega_0 = 2\pi \)
3. \( b = 4 \) and \( \omega_0 = \pi/2 \)
4. \( b = 4 \) and \( \omega_0 = 2\pi \)
5. none of the above
**Properties of Fourier Transforms**

Find a general scaling rule.

Let \( f_2(t) = f_1(at) \) where \( a > 0 \).

\[
F_2(\omega) = \int_{-\infty}^{\infty} f_1(at)e^{-j\omega t} dt
\]

Let \( \tau = at \). Then \( d\tau = adt \).

\[
F_2(\omega) = \int_{-\infty}^{\infty} f_1(\tau)e^{-j\omega \tau/a} \frac{1}{a} d\tau = \frac{1}{a} F_1 \left( \frac{\omega}{a} \right)
\]

Stretching time compresses frequency and increases amplitude (preserving area).

---

**Moments**

The value of \( f(0) \) is the integral of \( F(\omega) \) divided by \( 2\pi \).

\[
f(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) d\omega
\]

---

**Stretching Time**

Stretching time compresses frequency and increases amplitude (preserving area).

---

**Compressing Time to the Limit**

Alternatively, we could compress time while keeping area = 1.

\[
f(t) \quad F(\omega) = \frac{\sin \omega/2}{\omega/2} \frac{1}{2\pi}
\]

In the limit, the pulse has zero width but area 1!

We represent this limit with the delta (or impulse) function: \( \delta(t) \).

---

**Math With Impulses**

Although physically unrealizable, the impulse (a.k.a. Dirac delta) function is useful as a mathematically tractable approximation to a very brief signal.

Example 1: Find the Fourier transform of a unit impulse function.

\[
F(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t} dt
\]

Since \( \delta(t) \) is zero except near \( t=0 \), only values of \( e^{-j\omega t} \) near \( t=0 \) are important. Because \( e^{-j\omega t} \) is a smooth function of \( t \), \( e^{-j\omega t} \) can be replaced by \( e^{-j\omega 0} \).

\[
F(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t} dt = 1
\]

This matches our previous result which was based explicitly on a limit. Here the limit is implicit.
Math With Impulses

Although physically unrealizable, the impulse (a.k.a. Dirac delta) function is useful as a mathematically tractable approximation to a very brief signal.

Example 2: Find the function whose Fourier transform is an impulse.

\[
\mathcal{F}\{f(t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} \, d\omega
\]

Notice the similarity to the previous result:

\[
\mathcal{F}\{f(t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} \, d\omega
\]

These relations are **duals** of each other.

- A constant in time consists of a single frequency at \( \omega = 0 \).
- An impulse in time contains components at all frequencies.

Relation Between Fourier Series and Fourier Transforms

If a periodic signal \( f(t) = f(t+T) \) has a Fourier series representation, then it can also be represented by an equivalent Fourier transform.

\[
e^{j\omega o T} \overset{\text{CTFS}}{\longleftrightarrow} 2\pi \delta(\omega - \omega o)
\]

\[
f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} F[k] e^{j2\pi k T} \quad \overset{\text{CTFS}}{\longleftrightarrow} \quad F[k]
\]

Each term in the Fourier series is replaced by an impulse in the Fourier transform.

Relation between Fourier Transform and Fourier Series

Each Fourier series term is replaced by an impulse in the Fourier transform.

\[
f(t) = \sum_{k=-\infty}^{\infty} f_p(t - kT)
\]

\[
\mathcal{F}\{f(t)\} = \sum_{k=-\infty}^{\infty} F[k] \delta(\omega - \frac{2\pi}{T} k)
\]

\[
F(\omega) = \sum_{k=-\infty}^{\infty} 2\pi F[k] \delta(\omega - \frac{2\pi}{T} k)
\]

Summary

Fourier series and transforms are similar: both represent signals by their frequency content.

**Continuous-Time Fourier Series**

\[
F[k] = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-j\omega o t} \, dt
\]

\[
f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} F[k] e^{j2\pi k T} \quad \text{where } \omega o = \frac{2\pi}{T}
\]

**Continuous-Time Fourier Transform**

\[
F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt
\]

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \, d\omega
\]

Next time: Fourier Transform for discrete-time signals.