Signals and Series
- Relations between time and frequency.
- Fourier series for discontinuous functions.
- Fourier analysis of a vibrating string.

Lab 1 check-in due before Friday at 4pm.

Office Hours
after recitation in the recitation room
Thursdays 7-9pm (in 36-144)
Fridays 1-4pm (in 36-144)

February 3, 2022

Last Time
Signals are functions that are used to convey information. Example: a musical sound can be represented as a function of time.

Although this time function is a complete description of the sound, it does not expose many of the important properties of the sound.

Last Time
Time functions do a poor job of conveying consonance and dissonance.

Harmonic structure conveys consonance and dissonance better.

Example of Analysis
Find the Fourier series coefficients for the following triangle wave:

\[ f(t) = \begin{cases} 
1 & \text{for } 0 \leq t < 1 \\
0 & \text{for } 1 \leq t < 2 
\end{cases} \]

This results in the following expressions for the Fourier series coefficients:

\[
c_0 = \frac{1}{T} \int_T f(t) \, dt
\]
\[
c_k = \frac{2}{T} \int_T f(t) \cos(k\omega_0 t) \, dt; \quad k = 1, 2, 3, \ldots
\]
\[
d_k = \frac{2}{T} \int_T f(t) \sin(k\omega_0 t) \, dt; \quad k = 1, 2, 3, \ldots
\]

where \( \omega_0 = \frac{2\pi}{T} \) represents the fundamental frequency.

Basis functions:

Harmonic structure conveys consonance and dissonance better.

Example of Analysis
Find the Fourier series coefficients for the following triangle wave:

\[ f(t) = \int_{t+2} f(t+2) \]

This results in the following expressions for the Fourier series coefficients:

\[
c_0 = \frac{1}{T} \int_T f(t+2) \, dt
\]
\[
c_k = \frac{2}{T} \int_T f(t+2) \cos(k\omega_0 t) \, dt = \frac{1}{2} \int_0^2 f(t) \cos(kt) \, dt = \frac{1}{2}
\]
\[
d_k = \frac{2}{T} \int_T f(t+2) \sin(k\omega_0 t) \, dt = 2 \int_0^1 t \cos(kt) \, dt = \left\{ \begin{array}{ll}
\frac{4}{k^2} & \text{if } k \text{ odd} \\
0 & \text{if } k = 2, 4, 6, \ldots
\end{array} \right.
\]
\[
d_k = 0 \quad \text{(by symmetry)}
\]
**Example of Synthesis**

Generate \( f(t) \) from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

\[
f(t) = c_0 + \sum_{k=1}^{\infty} \left( c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t) \right) = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t) \]

\[
f(t) = \frac{1}{2} - \sum_{k=1}^{99} \frac{4}{\pi^2 k^2} \cos(k\pi t) \]

The synthesized function approaches original as number of terms increases.

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**Fourier Analysis of a Square Wave**

Find the Fourier series coefficients for the following square wave:

\[
f(t) = f(t+2)
\]

\(T = 2\)

\(\omega_0 = \frac{2\pi}{T} = \pi\)

\(c_0 = \frac{1}{T} \int_{-1}^{1} f(t) \, dt = \frac{1}{2} \int_{0}^{2} f(t) \, dt = \frac{1}{2}\)

\(c_k = \frac{2}{T} \int_{0}^{T} f(t) \cos(k\omega_0 t) \, dt \quad \int_{-1}^{1} \cos(k\pi t) \, dt = \frac{\sin(k\pi t)}{k\pi} \bigg|_{-1}^{1} = 0 \) for \(k = 1, 2, 3, \ldots\)

\(d_k = \frac{2}{T} \int_{0}^{T} f(t) \sin(k\omega_0 t) \, dt \quad \int_{-1}^{1} \sin(k\pi t) \, dt = -\frac{\cos(k\pi t)}{k\pi} \bigg|_{-1}^{1} = \begin{cases} \frac{2}{k\pi} & k = 1, 3, 5, \ldots \text{ otherwise} \\ 0 & \end{cases} \)

The synthesized function approaches original as number of terms increases.

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**Fourier Synthesis of a Square Wave**

Generate \( f(t) \) from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

\[
f(t) = c_0 + \sum_{k=1}^{\infty} \left( c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t) \right) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{k\pi} \sin(k\pi t) \
\]

\[
f(t) = \frac{1}{2} + \sum_{k=1}^{99} \frac{2}{k\pi} \sin(k\pi t) \
\]

The synthesized function approaches original as number of terms increases.

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**Convergence of Fourier Series**

If there is a step discontinuity in \( f(t) \) at \( t = t_0 \), then the Fourier series for \( f(t_0) \) converges to the average of the limits of \( f(t) \) as \( t \) approaches \( t_0 \) from the left and from the right.

Let \( f_K(t) \) represent the partial sum of the Fourier series using just \( N \) terms:

\[
f_K(t) = a_0 + \sum_{k=1}^{K} \left( c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t) \right) \
\]

As \( K \to \infty \),
- the maximum difference between \( f(t) \) and \( f_K(t) \) converges to \( \approx 9\% \) of \( |f(t_0^-) - f(t_0^+)\) and
- the region over which the absolute value of the difference exceeds any small number \( \epsilon \) shrinks to zero.

We refer to this type of overshoot as **Gibb’s Phenomenon**.

**So who was right?** Fourier or Lagrange?

Both. The series representation of a discontinuous function converges, but in an unusual way.
Properties of Fourier Series

How do changes in signals affect their frequency representation?

→ investigate properties of Fourier representations

Properties of Fourier Series: Scaling Time

Find the Fourier series coefficients for the following square wave:

\[ g(t) = g(t+1) \]

We could repeat the process used to find the Fourier coefficients for \( f(t) \).

Alternatively, we can take advantage of the relation between \( f(t) \) and \( g(t) \):

\[ g(t) = f(2t) \]

Scaling Time

We already know the Fourier series expansion of \( f(t) \):

\[ f(t) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{k\pi} \sin(k\pi t) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{k\pi} \sin(k\omega_o t) \]

\[ c_k = \begin{cases} \frac{1}{k\pi} & \text{if } k=0 \\ \frac{2}{k\pi} & \text{if } k=1,3,5,\ldots \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad d_k = 0 \]

where \( \omega_o = \frac{2\pi}{T} = \frac{\pi}{2} = \pi \).

Since \( g(t) = f(2t) \) it follows that

\[ g(t) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{k\pi} \sin(2k\pi t) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{k\pi} \sin(k\omega_1 t) \]

The Fourier series coefficients for \( g(t) \) are thus identical to those of \( f(t) \). Only the fundamental frequency has changed, from \( \omega_o = \pi \) to \( \omega_1 = 2\pi \).

Scaling Time

Plot the Fourier series coefficients on a frequency scale.

Compressing the time axis has stretched the \( \omega \) axis.

What is the Effect of Shifting Time?

Assume that \( f(t) \) is periodic in time with period \( T \):

\[ f(t) = f(t+T) \]

Let \( g(t) \) represent a version of \( f(t) \) shifted by half a period:

\[ g(t) = f(t - T/2) \]

How many of the following statements correctly describe the effect of this shift on the Fourier series coefficients.

- cosine coefficients \( c_k \) are negated
- sine coefficients \( d_k \) are negated
- odd-numbered coefficients \( c_1, d_1, c_3, d_3, \ldots \) are negated
- sine and cosine coefficients are swapped: \( c_k \rightarrow d_k \) and \( d_k \rightarrow c_k \)
Why Focus on Fourier Series?
What’s so special about sines and cosines?
Sinusoidal functions have interesting mathematical properties.
→ harmonically related sinusoids are orthogonal to each other over \([0, T]\).

Orthogonality: \(f(t)\) and \(g(t)\) are orthogonal over \(0 \leq t \leq T\) if
\[
\int_T f(t)g(t)\,dt = 0
\]

Example: Calculate this integral for the \(k\)th and \(l\)th harmonics of \(\cos(\omega_0 t)\).
\[
\int_T \cos(k\omega_0 t)\cos(l\omega_0 t)\,dt
\]

We can use trigonometry to express the product of the two cosines as the sum of cosines at the sum and difference frequencies:
\[
\int_T \left(\frac{1}{2} \cos((k+l)\omega_0 t) + \frac{1}{2} \cos((k-l)\omega_0 t)\right)\,dt
\]
The sum and difference frequencies are also harmonics of \(\omega_0\), so their integral over \(2\pi/T\) is zero (provided \(k \neq l\)).

Physical Example: Vibrating String
A taut string supports wave motion.

The speed of the wave depends on the tension on and mass of the string.

Physical Example: Vibrating String
The wave will reflect off a rigid boundary.

The amplitude of the reflected wave is opposite that of the incident wave.

Physical Example: Vibrating String
Reflections can interfere with excitations.

The interference can be constructive or destructive depending on the frequency of the excitation.

Physical Example: Vibrating String
We get constructive interference if round-trip travel time equals the period.

Round-trip travel time = \(2L/v\) = \(T\)
\[
\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2L/v} = \frac{\pi v}{L}
\]
Physical Example: Vibrating String
We also get constructive interference if round-trip travel time is $2T$.

$x = 0$  
$x = L$

Round-trip travel time $= \frac{2L}{v} = 2T$

$\omega = \frac{2\pi}{T} = \frac{2\pi}{L/v} = 2\omega_o$

Physical Example: Vibrating String
In fact, we also get constructive interference if round-trip travel time is $kT$.

$x = 0$  
$x = L$

Round-trip travel time $= \frac{2L}{v} = kT$

$\omega = \frac{2\pi}{T} = \frac{2\pi}{2L/kv} = \frac{k\pi v}{L} = k\omega_o$

Only certain frequencies (harmonics of $\omega_o = \pi v/L$) persist. This is the basis of stringed instruments.

Physical Example: Vibrating String
More complicated motions can be expressed as a sum of normal modes using Fourier series. Here the string is “plucked” at $x = l$.

$x = 0$  
$x = L$

Physical Example: Vibrating String

$L/2$

no even harmonics

$L/3$

no multiples of 3

$L/4$

no multiples of 4

$L/(2\pi)$

all harmonics

Differences in harmonic structure generate differences in timbre.

Summary
- We examined the convergence of Fourier series.
  - Functions with discontinuous slopes well represented.
  - Functions with discontinuous values generate ripples → Gibb’s phenomenon.
- We investigated several properties of Fourier series.
  - scaling time
  - shifting time
  - We will find that there are many others
- We saw how Fourier series are useful for modeling a vibrating string.