6.003 (Spring 2021) *12 May 2021*

Quiz 2 Practice 2: Solutions

1. Transforms

Part a.

Each cosine gives us a pair of deltas symmetric about $k = 0$:

$$X_1[k] = \delta[k+3] + 2\delta[k+2] + 2\delta[k-2] + \delta[k-3]$$

Part b.

Direct application of the analysis formula gives:

$$X_2[k] = \frac{1}{10}\left(1 + e^{-j\frac{2\pi}{5}k}\right)$$

Part c.

If we consider a related signal $x_{3,s}(t)$ that corresponds to a rectangular pulse of the height and width but symmetric about $t = 0$, then we would have:

$$x_{3,s}(t) = \begin{cases} 1 & \text{if } -3/2 \le t \le 3/2 \\ 0 & \text{otherwise} \end{cases}$$

$$X_{3,s}(\omega) = \frac{2\sin\left(\frac{3\omega}{2}\right)}{\omega}$$

We can compute x_3 by shifting $x_{3,s}$ to the right by $1/2$ seconds, so, by the time shift property:

$$X_3(\omega) = e^{-j\omega/2}X_{3,s}(\omega) = e^{-j\omega/2}\left(\frac{2\sin\left(\frac{3\omega}{2}\right)}{\omega}\right)$$

Part d.

Direct application of the DTFT analysis equation yields:

$$X_4(\Omega) = 2\cos(3\Omega) + 2j\sin(\Omega)$$

2. 2D Convolution

$(x_1 \circledast x_0)$ matches graph **F**

$(x_2 \circledast x_0)$ matches graph **D**

$(x_3 \circledast x_0)$ matches graph **L**

$(x_4 \circledast x_0)$ matches graph **C**

$(x_5 \circledast x_0)$ matches graph **G**

$(x_6 \circledast x_0)$ matches graph **K**

$(x_7 \circledast x_0)$ matches graph **I**

$(x_8 \circledast x_0)$ matches graph **A**

$(x_9 \circledast x_0)$ matches graph **E**

$(x_{10} \circledast x_0)$ matches graph **J**

3. Filtering

There are a few keys things to consider here. One is the difference between shapes (circle vs square vs Gaussian), and another has to do with the size of the nonzero portion in DFT space.

As we have seen before, any shapes with a sharp edge in DFT space (filters A, C, D, E, and F) will result in ringing artifacts in the output. So those must be associated with 1, 2, 4, 5, 7, or 8).

When considering the spatial-domain equivalents of something with a circular shape in the DFT space, we will see a radially-symmetric (but still sinc-like) shape in space. So convolving with that kind of shape should lead to artifacts that have a similar kind of symmetry. By contrast, the square-like shapes will have ringing artifacts that show up most prominently in the purely-vertical and purely-horizontal directions.

This is the difference between, for example, 1 and 4, versus 2 and 5. Images 1 and 4 have the kind of symmetric artifacts we would expect to arise from the circular filters, whereas in 2 and 5, the artifacts are most noticeable in the vertical and horizontal directions. This difference is most noticeable in the corners of the images, and in diagonal directions from any sharp edge in the original.

We also notice a big difference in terms of the size of the filter. Filters with a smaller region of nonzero values around the origin will be blurrier (they let in only the lower frequencies, whereas the bigger shapes let more higher frequencies pass). This is the difference between 1 and 2 (sharper), versus 4 and 5 (blurrier).

Filters A and C are interesting in that they are asymmetric (i.e., they blur more along one axis than along the other). A should be blurrier in the vertical direction than the horizontal direction, and *vice versa* for C.

Finally, the Gaussian filter B should result in fewer ringing artifacts but should still blur its output. This narrows the choices to 9 and 12, but, given B's similar size to filter D, the sharper of those two is more appropriate.

Taking all of this into consideration, we find:

Filter A matches signal y_8

Filter B matches signal y_{12}

Filter C matches signal y_7

Filter D matches signal y_1

Filter E matches signal y_2

Filter F matches signal y_4

4. Bandpass Filter

Part 1

We have $\Omega_c = 0.12\pi$. Since $\Omega = \frac{2\pi f}{f_s}$, we have $\frac{0.12\pi=1200\pi}{f_s}$, which implies that we need $f_s = 10000$ samples / second to accomplish this goal.

Next, we'll find $h[n]$. Here, it may be easiest to look at $H(\Omega)$ as being the difference of two signals: $H(\Omega) = H_1(\Omega) - H_2(\Omega)$, where

$$H_1(\Omega) = \begin{cases} 1 & \text{if } |\Omega| \le 0.15\pi \\ 0 & \text{otherwise} \end{cases}$$

$$H_2(\Omega) = \begin{cases} 1 & \text{if } |\Omega| \le 0.09\pi \\ 0 & \text{otherwise} \end{cases}$$

Plugging into the synthesis formula, we find that $h_1[n] = \frac{\sin(0.15\pi n)}{\pi n}$ and $h_2[n] = \frac{\sin(0.09\pi n)}{\pi n}$
So, overall, we have:

$$h[n] = h_1[n] - h_2[n] = \frac{\sin(0.15\pi n) - \sin(0.09\pi n)}{\pi n}$$

Alternatively, notice that we can represent $H(\Omega)$ as $(H_3 * H_4)(\Omega)$, where

$$H_3(\Omega) = \begin{cases} 1 & \text{if } |\Omega| \le 0.03\pi \\ 0 & \text{otherwise} \end{cases}$$

$$H_4(\Omega) = \delta(\Omega - 0.12\pi) + \delta(\Omega + 0.12\pi)$$

Therefore, we have:

$$h[n] = h_3[n] \times h_4[n] = \left(\frac{\sin(0.03\pi n)}{\pi n}\right)(2\cos(0.12\pi n))$$

Although it may not look like it, the two expressions we've given for $h[n]$ above are exactly equivalent.

Part 2

When we sample $H(\Omega)$ at integer multiples of $\frac{2\pi}{10}$, all ten values are 0.

$$H_{10}[k] = H(\Omega)\Big|_{\Omega=0,\frac{2\pi}{10},\frac{4\pi}{10},\ldots,2\pi} = 0$$
$$h_{10}[n] = 0$$

When we instead sample at integer multiples of $\frac{2\pi}{20}$, we have two nonzero values, associated with $\Omega = \pm 0.1\pi$. That is, $H_{20}[k] = \delta[k-1] + \delta[k+1]$, which, when $N = 20$, implies that $h_{20}[n] = 2\cos(0.1\pi n)$

$$H_{20}[k] = H(\Omega)\Big|_{\Omega=0,\frac{2\pi}{20},\frac{4\pi}{20},\ldots,2\pi} = \begin{cases} 1 & \text{if } |k| = 1 \\ 0 & \text{otherwise} \end{cases}$$
$$h_{20}[n] = 2\cos\left(\frac{2\pi kn}{N}\right) = 2\cos(0.10\pi n)$$

When we instead sample at integer multiples of $\frac{2\pi}{50}$, we still only have two nonzero value, no associated with $\Omega = \pm 0.12\pi$. Thus, $h_{50}[n] = 2\cos(0.12\pi n)$

$$H_{50}[k] = H(\Omega)\Big|_{\Omega=0,\frac{2\pi}{50},\frac{4\pi}{50},\ldots,2\pi} = \begin{cases} 1 & \text{if } |k| = 3 \\ 0 & \text{otherwise} \end{cases}$$
$$h_{50}[n] = 2\cos\left(\frac{2\pi kn}{N}\right) = 2\cos(0.12\pi n)$$

We can follow similar logic for H_{100}, though there are now three pairs of deltas:

$$H_{100}[k] = H(\Omega)\Big|_{\Omega=0,\frac{2\pi}{100},\frac{4\pi}{100},\ldots,2\pi} = \begin{cases} 1 & \text{if } |k| = 5, 6, \text{or } 7 \\ 0 & \text{otherwise} \end{cases}$$
$$h_{100}[n] = 2\cos(0.10\pi n) + 2\cos(0.12\pi n) + 2\cos(0.14\pi n)$$

Part 3

$H_N[k]$ is a sampled version of $H(\Omega)$: $H_N[k] = H\left(\frac{2\pi k}{N}\right)$. Therefore, $h_N[n]$ is an aliased version of $h[n]$:

$$h_N[n] = N \sum_{m=-\infty}^{\infty} h[n - mN]$$

5. 2D DFT

Part 1

The fact that we have a single dot in the spatial domain makes the transform relatively easy to find:

$$F_1[k_r, k_c] = \frac{1}{256} e^{-j\left(\frac{6\pi}{16}k_r + \frac{4\pi}{16}k_c\right)}$$

Importantly, note that the magnitude of *every* value is $\frac{1}{256}$. Thus, the answers to the first two questions are $\frac{1}{256}$.

For the third and fourth parts of the problem, we are looking for the real and imaginary parts, respectively, of:

$$F_1[4,4] = \frac{1}{256} e^{-j\left(\frac{6\pi}{16}4 + \frac{4\pi}{16}4\right)} = \frac{1}{256} e^{-j\left(\frac{3\pi}{2} + \pi\right)} = \frac{1}{256} e^{-j\frac{5\pi}{2}} = \frac{1}{256} e^{-j\frac{\pi}{2}} = \frac{-j}{256}$$

Thus, the real part is 0 and the imaginary part is $\frac{-1}{256}$

Part 2

Here, we can think of $f_2[r,c]$ as being made up of two smaller signals: $f_2[r,c] = f_{2a}[r,c] + f_{2b}[r,c]$, where

$$f_{2a}[r,c] = \delta[r-3]$$

$$f_{2b}[r,c] = \delta[c-2]$$

We can then compute the 2D DFT's of each of these pieces:

$$F_{2a}[k_r, kc] = \frac{1}{16}\delta[k_c] e^{-j\frac{6\pi k_r}{16}}$$

$$F_{2a}[k_r, kc] = \frac{1}{16}\delta[k_r] e^{-j\frac{4\pi k_c}{16}}$$

Thus, we have:

$$F_2[k_r, k_c] = F_{2a}[k_r, k_c] + F_{2b}[k_r, k_c] = \frac{\delta[k_c] e^{-j\frac{6\pi k_r}{16}} + \delta[k_r] e^{-j\frac{4\pi k_c}{16}}}{16}$$

Importantly, note that this function has a value of 0 for anything that is not at $k_r = 0$ or $k_c = 0$. So the second and fourth questions are going to be zero.

For the first question, note that when $k_r = 3$ and $k_c = 0$, the $\delta[k_r = 3] = 0$, so we only have the one delta complex exponential contributing, and so the magnitude is $1/16$.

Then, for the third question, we have:

$$\text{Re}\left(F_2[5,0]\right) = \text{Re}\left(e^{-j\frac{30\pi}{16}}\right) = \frac{\cos\left(\frac{30\pi}{16}\right)}{16} = \frac{\cos\left(\frac{\pi}{8}\right)}{16}$$