6.003 (Spring 2021) *12 May 2021*

Quiz 2 Practice 1: Solutions

1. Short Answers

Part a.

$$x_1(t) = 2\cos\left(\frac{\pi}{12}t\right) + \cos\left(\frac{7\pi}{12}t\right)$$

$$T_1 = \frac{2\pi}{\pi/12} = 24$$

Part b.

$$\begin{aligned} x_2[n] &= 3\sin(\Omega_o n) - 4\sin^3(\Omega_o n) \\ &= 3\left(\frac{e^{j\Omega_o n} - e^{-j\Omega_o n}}{j2}\right) - 4\left(\frac{e^{j\Omega_o n} - e^{-j\Omega_o n}}{j2}\right)^3 \\ &= 3\left(\frac{e^{j\Omega_o n} - e^{-j\Omega_o n}}{j2}\right) + \left(\frac{e^{j3\Omega_o n} - 3e^{j\Omega_o n} + 3e^{-j\Omega_o n} - e^{-j3\Omega_o n}}{j2}\right) \\ &= \left(\frac{e^{j3\Omega_o n} - e^{-j3\Omega_o n}}{j2}\right) \\ &= \sin(3\Omega_o n) \end{aligned}$$

Just one non-zero term (c_3) is required.

Part c.

$$X_3(\omega) = \int_0^\infty e^{-2(t-3)} e^{-j\omega t}\, dt$$

$$X_3(0) = \int_0^\infty e^{-2(t-3)} e^0\, dt = \left.\frac{e^{-2(t-3)}}{-2}\right|_0^\infty = \frac{1}{2}e^6$$

Part d.

$$x_4[n] = \delta[n] + 2\delta[n-5]$$

$$\begin{aligned} X_4[k] &= \frac{1}{10}\sum_{n=0}^{9} x_4[n] e^{-j\frac{2\pi k}{10}n} \\ &= \frac{1}{10}\left(1 + 2e^{-j\frac{2\pi k}{10}5}\right) \\ &= \frac{1}{10}\left(1 + 2e^{-j\pi k}\right) \end{aligned}$$

$$X_4[2] = \frac{3}{10}$$

Part e.

$x_5[n]$ must be one of the basis functions of the DCT when $N = 8$.

$$X_C[k] = \frac{1}{8} \sum_{n=0}^{7} x_5[n] \cos\left(\frac{\pi k}{8}\left(n + \frac{1}{2}\right)\right)$$

Therefore $k = 5$, $\phi = \frac{5\pi}{16}$, and $A = 4$.

2. Impulsive Images

Part a.

The delta at $(0,0)$ gives us a constant $2/RC$ in the frequency domain, and the pair of deltas at $(-2,1)$ and $(2,-1)$ gives us a cosine of amplitude $2/RC$. Thus, the maximum value in the image is $4/RC$, and the minimum value is 0. This cosine should go through 2 cycles vertically over the course of the image, and 1 cycle horizontally. The matching image is **G**.

Part b.

Here, we have the multiplication of two signals in the spatial domain, which will correspond to the convolution of those signals in the frequency domain. The first signal has a DFT of $\delta[k_c](2/RC + 2\cos(2\pi k_r/R)/RC)$, and the second has a DFT of $\delta[k_r](2j\sin(2\pi k_c/C)/RC)$. Convolving these together, we should end up with a signal that looks like a sine wave horizontally (which goes through a single cycle over the course of the image) and a cosine wave vertically (which also goes through a single cycle over the course of the image). The matching image is **J**.

Part c.

We can compute $F_3[k_r, k_c]$ by noticing that the delta at $(0,0)$ gives us a constant, and the two other pairs each give us a sine wave:

$$F_3[k_r, k_c] = \frac{4}{RC} - \frac{2j}{RC}\left(\sin\left(\frac{4\pi}{C}k_c\right)\right) + \frac{2j}{RC}\left(\sin\left(\frac{4\pi}{R}k_r\right)\right)$$

It is maybe a little bit difficult to predict what this will look like, espcially since this expression will never go to zero (the minumim magnitude will be $4/RC$, and the maximum magnitude will be $4\sqrt{2}/RC$). But we can see a few things. One is that if we hold k_r constant, we should see something like a sine wave as we move across the row (it should go through two full cycles over the course of the image), and then if we change k_r (so as to look across a different row), we should still see a sine wave, just offset by some amount).

The matching image is **P**.

3. Inverse DTFT

Looking at the graph, we can see that this corresponds to

$$F(\Omega) = \cos\left(\frac{3}{2}\Omega\right) e^{-j(\Omega/2)} = \left(\frac{e^{j3\Omega/2} + e^{-j3\Omega/2}}{2}\right) e^{-j(\Omega/2)} = \frac{e^{-j2\Omega} + e^{j\Omega}}{2}$$

Then, since we know that $F(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$, we can see that:

$$f[n] = \frac{\delta[n-2]}{2} + \frac{\delta[n+1]}{2} = \begin{cases} 1/2 & \text{if } n \in \{-1, 2\} \\ 0 & \text{otherwise} \end{cases}$$

4. Number Transforms

Digits with rounded elements will show some sort of radially-symmetric pattern in the DFT. Digits with straight lines will show sincs (roughly in the opposite direction) in the DFT. By matching these patterns, we find:

Graph a $\rightarrow$ Digit **4**

Graph b $\rightarrow$ Digit **3**

Graph c $\rightarrow$ Digit **5**

Graph d $\rightarrow$ Digit **1**

Graph e $\rightarrow$ Digit **2**

5. Continuous-Time Transforms

We are given:

$$f_1(t) = \begin{cases} a & \text{if } |t| < b \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(t) = f_1(t) + c\delta(t)$$

$$f_3(t) = \frac{T}{2\pi} \sum_{m=-\infty}^{\infty} f_2(t + mT)$$

We can find $F_1(\omega)$ directly to find a familiar sinc pattern:

$$F_1(\omega) = \int_{-\infty}^{\infty} f_1(t)e^{-j\omega t}\mathrm{d}t = \int_{-b}^{b} ae^{-j\omega t}\mathrm{d}t = \frac{2a\sin(\omega b)}{\omega}$$

By linearity, $F_2(\omega) = F_1(\omega) + c$

$f_3(t)$ is a periodically-extended version of $f_2(t)$, repeated every T seconds.

Part a.

Here, we know we are looking for a sinc function that is offset by 1, so A, B, C, or D are all plausible. However, the height and zero-crossing of the sinc function can help us differentiate between the graphs.

In this case, we expect the first zero-crossing of the sinc (or the first time this graph crosses 1) to occur when the argument to the sine, ωb, is equal to pi. This occurs at $\omega = 2\pi$, which eliminates all possibilities except **B**.

Part b.

Here, we can use similar reasoning to part a. However, here, we see the first zero crossing should occur at $\omega = \pi$, which would be consistent with both graph A and graph C. We can tell these apart, though, by looking at the height of the sinc.

We expect the peak value (at $\omega = 0$) of the sinc alone (without the offset of 1) to be $2ab$, which, given the parameters here, will be 3. Thus, we expect that, with the offset, we will see a value of 4 at $\omega = 0$, which is consistent with graph **C**.

Part c.

Since $f_3(t)$ is periodic in T, we know that we will only need frequencies that are integer multiples of $\frac{2\pi}{T}$ in order to represent it. Thus, we know that the only possibilities are graphs I, J, K, or L.

Given the $T/2\pi$ scaling factor, the other values will be the same as the graph from part a., which is consistent with graph **J**.

Part d.

Using similar reasoning to part c, we find that the matching graph is **K**.

6. Transformational

Importantly, we can solve this problem without doing any complicated math, so long as we remember some properties of DFT's.

The key here is that, from the information provided, we know that $y[n] = \frac{1}{8}\left(x_w \circledast w\right)[n]$

And we know, within our analysis window, that $w[n] = \{1, 1, 1, 1, 1, 1, 0, 0\}$ and $x_w[n] = \{0, 1, 2, 3, 4, 5, 0, 0\}$.

By superposition (or some other method), we can find that $(x_w \circledast w)[n] = \{12, 10, 8, 6, 10, 15, 15, 14\}$

So $y[n] = \left\{\frac{3}{2}, \frac{5}{4}, 1, \frac{3}{4}, \frac{5}{4}, \frac{15}{8}, \frac{15}{8}, \frac{7}{4}\right\} = \{1.5, 1.25, 1, 0.75, 1.25, 1.875, 1.875, 1.75\}$