6.003 (Spring 2021) *24 March 2021*

Quiz 1: Solutions

1. Sinusoids

Part a

This cosine looks like it goes through one period around every 10 samples, so $\Omega_1 \approx \frac{2\pi}{10}$.

The amplitude looks like 2, but we start at a negative peak; so we could either say $A_1 \approx 2$ and $\phi_1 \approx \pm\pi$ (or any $m\pi$, where m is an odd integer); or we could say $A_1 \approx -2$ and $\phi_1 \approx 0$ (or any integer multiple of 2π).

Part b

Here we have $x_2[n] = a_2 e^{j\Omega_2 n} + a_2^* e^{-j\Omega_2 n} = 2|a_2|\cos(\Omega_2 n + \angle a_2)$.

It looks like the amplitude of this wave is about 1, so $|a_2| \approx 1/2$.

The cosine seems to go through one full cycle every 15 samples, so $\Omega_2 \approx 2\pi/15$.

It also looks like it has been shifted to the right by 3/4 of a full cycle (or, equivalently, to the left by 1/4 of a full cycle), so $\angle a_2 \approx 2\pi/4 = \pi/2$.

Part c

We can express $x_3[n] = c_3 \cos \Omega_3 n + d_3 \sin \Omega_3 n$ as

$$\begin{aligned} x_3[n] &= c_3 \left(\frac{e^{j\Omega_3 n} + e^{-j\Omega_3 n}}{2}\right) + d_3 \left(\frac{e^{j\Omega_3 n} - e^{-j\Omega_3 n}}{2j}\right) \\ &= \frac{1}{2}(c_3 - jd_3)e^{j\Omega_3 n} + \frac{1}{2}(c_3 + jd_3)e^{-j\Omega_3 n} \\ &= \text{Re}\left((c_3 - jd_3)e^{j\Omega_3 n}\right) \end{aligned}$$

Since the period of $x_3[n]$ is around 20, it follows that $\Omega_3 \approx \frac{2\pi}{20}$

The peak amplitude of $x_3[n]$ is approximately 1, so $|c_3 - jd_3| = \sqrt{c_3^2 + d_3^2} = 1$.

The first peak of $x_3[n]$ occurs at about 1/8 of a cycle. So the angle of $c_3 - jd_3$ must be approximately $-\pi/4$, meaning $c_3 \approx d_3$. Therefore,

$$c_3 \approx d_3 \approx \frac{1}{\sqrt{2}}$$

2. Fourier Transforms

Part 1

Note that $b = X(\omega = 0)$, or, said another way, b is the DC component of $x(\cdot)$, which will be equal to the total "area under the curve" of $x(\cdot)$, so $b = (t_2 - t_1)a$.

Part 2

Here, we probably want to solve for $X(\omega)$ more generally, to get a sense of how the function behaves.

It will be simpler to consider a related function $x_c(\cdot)$ that is a rectangular pulse of the same width and height, but centered around $t = 0$ rather than being offset. In that case, since we have $x(t) = x_c\left(t - (t_1 + t_2/2)\right)$, we know that:

$$X(\omega) = e^{-j\left(\frac{t_1+t_2}{2}\omega\right)} X_c(\omega).$$

We can then solve for $X_c(\omega)$:

$$X_c(\omega) = \int_{-\infty}^{\infty} x_c(t)e^{-j\omega t}\mathrm{d}t = \int_{-(t_2-t_1)/2}^{(t_2-t_1)/2} ae^{-j\omega t}\mathrm{d}t = a\int_{-(t_2-t_1)/2}^{(t_2-t_1)/2} e^{-j\omega t}\mathrm{d}t$$
$$= -\frac{a}{j\omega}\, e^{-j\omega t}\Big|_{t=-(t_2-t_1)/2}^{(t_2-t_1)/2} = -\frac{a}{j\omega}\left(e^{-j\omega(t_2-t_1)/2} - e^{j\omega(t_2-t_1)/2}\right) = \frac{2a\sin\left(\frac{t_2-t_1}{2}\omega\right)}{\omega}$$

So, all things considered, we have:

$$X(\omega) = e^{-j\left(\frac{t_1+t_2}{2}\omega\right)} X_c(\omega) = e^{-j\left(\frac{t_1+t_2}{2}\omega\right)} \frac{2a\sin\left(\frac{t_2-t_1}{2}\omega\right)}{\omega}$$

From this, we can see that $|X(\Omega)| = |X_c(\Omega)| = \left|\frac{2a\sin\left(\frac{t_2-t_1}{2}\omega\right)}{\omega}\right|$

The value ω_1 is where this function first crosses zero, which will happen when the argument to sin is π, so we have:

$$\left(\frac{t_2 - t_1}{2}\right)\omega_1 = \pi \qquad \Rightarrow \qquad \omega_1 = \frac{2\pi}{t_2 - t_1}$$

Parts 3 and 4

From the work above, we also know that $\angle\{X(\Omega)\} = \angle\{X_c(\Omega)\} - \left(\frac{t_1+t_2}{2}\omega\right)$

So we should see a linear shape in the graph, with slope $-\frac{t_1+t_2}{2}$. However, we should also note that since $X_c(\Omega)$ switched signs every ω_1 radians, $\angle\{X_c(\Omega)\}$ was increasing (or decreasing) by π every ω_1 radians as well. So we not only to see a linear slope, but we also expect jumps upward every so often. The only graph that fits the bill is graph **F**.

The value c corresponds to the slope, which should be $-\frac{t_1+t_2}{2}$, as seen above.

3. Related Transforms

Part 1

$$f_2[n] = -f[-n] \qquad \Rightarrow \qquad F_2(\Omega) = -F(-\Omega)$$

Part 2

$$f_3[n] = f[n] - \delta[n] \qquad \Rightarrow \qquad F_3(\Omega) = F(\Omega) - 1$$

Part 3

$$f_4[n] = Asym\left\{f[n]\right\} = \frac{f[n] - f[-n]}{2} \qquad \Rightarrow \qquad F_4(\Omega) = j\text{Im}\left\{F(\Omega)\right\}$$

Part 4

$$f_5[n] = f[n+2] \qquad \Rightarrow \qquad F_5(\Omega) = e^{j2\Omega}F(\Omega)$$

Part 5

If we let:

$$f_s[n] = \begin{cases} f[n/3] & \text{if } n \text{ is divisible by 3} \\ 0 & \text{otherwise} \end{cases}$$

Then we have $f_6[n] = f_s[n] + f_s[n-1] + f_s[n+1]$, so:

$$F_6(\Omega) = F_s(\Omega)\left(1 + e^{j\Omega} + e^{-j\Omega}\right) = F_s(\Omega)\left(1 + 2\cos(\Omega)\right)$$

Since $F_s(\Omega) = F\left(3\Omega\right)$, we have:

$$F_6(\Omega) = \left(F\left(3\Omega\right)\right)\left(1 + 2\cos(\Omega)\right)$$

4. DFT

Part 1

We know that $k = f\frac{N}{f_s}$, so since $k = \pm 268 \approx \pm\frac{f}{2}$, we know that we must have $N \approx \frac{f_s}{2}$, so the only real possibility is $N = 1024$.

Part 2

Importantly, since the "A" note ($f = 1760\text{H}z$) is above the Nyquist rate, it is going to alias down to $f_{\text{apparent}} = f_s - 1760\text{H}z = 240\text{H}z$. Converting to k, we find $k = \pm 240\left(\frac{1024}{2000}\right) \approx \pm 123$.

We could also find this by first finding the expected k value for $f = 1760\text{H}z$ first:

$k = \pm\left(\frac{1024}{2000}\right)1760 \approx \pm 901$. But that's not in the range $[0, N/2]$, so it can't be the final answer. Since the coefficients are periodic in N, we need to find a value $1024m + 901$, where m is an integer, for which that value falls in the range $[0, 512]$. That happens to be $1024 - 901 = 123$.

Part 3

Since the "C" note shows up at $k = \pm 268$ and the "A" note (due to aliasing) shows up at $k = \pm 123$, the "A" note must correspond to the peaks that are closer to 0, i.e., k_1.

5. DTFS Components

The first six of these problems can be solved directly:

Part 0

$X_0[k] = 1/6$

This signal has a constant magnitude of 1/6 (graph **G**), a constant real part of 1/6 (graph **G**), and a constant imaginary part of 0 (graph **C**).

Looking ahead, importantly, since all of x_0, x_1, x_2, x_3, x_4, and x_5 are just time-shifted versions of each other, they should all have the same magnitude (graph **G**).

Part 1

$X_1[k] = \frac{1}{6}e^{-j\frac{2\pi}{6}k} = \frac{1}{6}\left(\cos\left(\frac{2\pi}{6}k\right) - j\sin\left(\frac{2\pi}{6}k\right)\right)$

The magnitude graph is still **G**, but now the real part looks like a cosine that goes through one period in $K = 6$ (graph **I**), and the imaginary part looks like an inverted sine that goes through one period in $K = 6$ (graph **D**).

Part 2

$X_2[k] = \frac{1}{6}e^{-j\frac{4\pi}{6}k} = \frac{1}{6}\left(\cos\left(\frac{4\pi}{6}k\right) - j\sin\left(\frac{4\pi}{6}k\right)\right)$

This is the same, but now the cosine and sine each go through two full periods in $K = 6$, so magnitude is still **G**, real part is **K**, and imaginary part is **E**.

Part 3

This one is interesting, in that $X_3[k]$ ends up being purely real:

$X_3[k] = \frac{1}{6}e^{-j\pi k} = \frac{1}{6}(-1)^k$

Magnitude is **G**, real part is **M**, and imaginary part is **C** (no imaginary part).

Part 4

$X_4[k] = \frac{1}{6}e^{-j\frac{8\pi}{6}k} = \frac{1}{6}e^{j\frac{4\pi}{6}k} = \frac{1}{6}\left(\cos\left(\frac{4\pi}{6}k\right) + j\sin\left(\frac{4\pi}{6}k\right)\right)$

This looks like $X_2[k]$, but with the imaginary part negated, so magnitude is **G**, real part is **K**, and imaginary part is **A**.

Part 5

$X_4[k] = \frac{1}{6}e^{-j\frac{10\pi}{6}k} = \frac{1}{6}e^{j\frac{2\pi}{6}k} = \frac{1}{6}\left(\cos\left(\frac{4\pi}{6}k\right) + j\sin\left(\frac{4\pi}{6}k\right)\right)$

This looks like $X_1[k]$, but with the imaginary part negated, so magnitude is **G**, real part is **I**, and imaginary part is **B**.

The next 6 parts all have simple relationships to the first 6 parts. In particular, note that for all i, $y_i[n] = 1 - x_i[n]$. Thus, by linearity, we have $Y_i[k] = \delta[k] - X_i[k]$, which we can use to simplify our analysis.

Since all of the signals y_i are just time-shifted versions of each other, all of the Y_i will have the same magnitude.

And for each $Y_i[k]$, we will have $\text{Re}(Y_i[0]) = 1 - \text{Re}(X_i[0])$, and for all other k, $\text{Re}(Y_i[k]) = -\text{Re}(X_i[k])$.

Similarly, for all k, $\text{Im}(Y_i[k]) = -\text{Im}(X_i[k])$.

Part 6

$Y_0[k] = \delta[k] - X_0[k] = \delta[k] = 1/6$

Magnitude **O**, real part **T**, imaginary part **C**.

Part 7

$Y_1[k] = \delta[k] - X_1[k]$

Magnitude **O**, real part **S**, imaginary part **B**.

Part 8

$Y_2[k] = \delta[k] - X_2[k]$

Magnitude **O**, real part **Q**, imaginary part **A**.

Part 9

$Y_3[k] = \delta[k] - X_3[k]$

Magnitude **O**, real part **N**, imaginary part **C**.

Part 10

$Y_4[k] = \delta[k] - X_4[k]$

Magnitude **O**, real part **Q**, imaginary part **E**.

Part 10

$Y_5[k] = \delta[k] - X_5[k]$

Magnitude **O**, real part **S**, imaginary part **D**.