Quiz 1: Solutions
1. Sinusoids

Part a
This cosine looks like it goes through one period around every 10 samples, so $\Omega_1 \approx \frac{2\pi}{10}$.
The amplitude looks like 2, but we start at a negative peak; so we could either say $A_1 \approx 2$ and $\phi_1 \approx \pm \pi$ (or any $m\pi$, where $m$ is an odd integer); or we could say $A_1 \approx -2$ and $\phi_1 \approx 0$ (or any integer multiple of $2\pi$).

Part b
Here we have
$$x_2[n] = a_2e^{j\Omega_2n} + a_2^*e^{-j\Omega_2n} = 2|a_2|\cos(\Omega_2n + \angle a_2).$$
It looks like the amplitude of this wave is about 1, so $|a_2| \approx 1/2$.
The cosine seems to go through one full cycle every 15 samples, so $\Omega_2 \approx 2\pi/15$.
It also looks like it has been shifted to the right by 3/4 of a full cycle (or, equivalently, to the left by 1/4 of a full cycle), so $\angle a_2 \approx 2\pi/4 = \pi/2$.

Part c
We can express
$$x_3[n] = c_3 \cos \Omega_3 n + d_3 \sin \Omega_3 n$$
as
$$x_3[n] = c_3 \left( \frac{e^{j\Omega_3n} + e^{-j\Omega_3n}}{2} \right) + d_3 \left( \frac{e^{j\Omega_3n} - e^{-j\Omega_3n}}{2j} \right)$$
$$= \frac{1}{2}(c_3 - jd_3)e^{j\Omega_3n} + \frac{1}{2}(c_3 + jd_3)e^{-j\Omega_3n}$$
$$= \text{Re} \left( (c_3 - jd_3)e^{j\Omega_3n} \right)$$
Since the period of $x_3[n]$ is around 20, it follows that $\Omega_3 \approx \frac{2\pi}{20}$
The peak amplitude of $x_3[n]$ is approximately 1, so $|c_3 - jd_3| = \sqrt{c_3^2 + d_3^2} = 1$.
The first peak of $x_3[n]$ occurs at about 1/8 of a cycle. So the angle of $c_3 - jd_3$ must be approximately $-\pi/4$, meaning $c_3 \approx d_3$. Therefore,
$$c_3 \approx d_3 \approx \frac{1}{\sqrt{2}}$$
2. Fourier Transforms

Part 1

Note that \( b = X(\omega = 0) \), or, said another way, \( b \) is the DC component of \( x(\cdot) \), which will be equal to the total 'area under the curve' of \( x(\cdot) \), so \( b = (t_2 - t_1)a \).

Part 2

Here, we probably want to solve for \( X(\omega) \) more generally, to get a sense of how the function behaves.

It will be simpler to consider a related function \( x_c(\cdot) \) that is a rectangular pulse of the same width and height, but centered around \( t = 0 \) rather than being offset. In that case, since we have \( x(t) = x_c(t - (t_1 + t_2)/2) \), we know that:

\[
X(\omega) = e^{-j\left(\frac{t_1 + t_2}{2}\omega\right)} X_c(\omega).
\]

We can then solve for \( X_c(\omega) \):

\[
X_c(\omega) = \int_{-\infty}^{\infty} x_c(t) e^{-j\omega t} dt = \int_{-(t_2-t_1)/2}^{(t_2-t_1)/2} ae^{-j\omega t} dt = a \int_{-(t_2-t_1)/2}^{(t_2-t_1)/2} e^{-j\omega t} dt
\]

\[
= -\frac{a}{j\omega} e^{-j\omega t \mid_{t=-(t_2-t_1)/2}^{(t_2-t_1)/2}} = -\frac{a}{j\omega} \left( e^{-j\omega(t_2-t_1)/2} - e^{j\omega(t_2-t_1)/2} \right) = \frac{2a \sin\left(\frac{t_2-t_1}{2}\omega\right)}{\omega}
\]

So, all things considered, we have:

\[
X(\omega) = e^{-j\left(\frac{t_1 + t_2}{2}\omega\right)} X_c(\omega) = e^{-j\left(\frac{t_1 + t_2}{2}\omega\right)} \frac{2a \sin\left(\frac{t_2-t_1}{2}\omega\right)}{\omega}
\]

From this, we can see that \( |X(\Omega)| = |X_c(\Omega)| = \frac{2a \sin\left(\frac{t_2-t_1}{2}\omega\right)}{\omega} \)

The value \( \omega_1 \) is where this function first crosses zero, which will happen when the argument to \( \sin \) is \( \pi \), so we have:

\[
\left(\frac{t_2 - t_1}{2}\right) \omega_1 = \pi \quad \Rightarrow \quad \omega_1 = \frac{2\pi}{t_2 - t_1}
\]

Parts 3 and 4

From the work above, we also know that \( \angle \{X(\Omega)\} = \angle \{X_c(\Omega)\} - \left(\frac{t_1 + t_2}{2}\omega\right) \)

So we should see a linear shape in the graph, with slope \(-\frac{t_1 + t_2}{2}\). However, we should also note that since \( X_c(\Omega) \) switched signs every \( \omega_1 \) radians, \( \angle \{X_c(\Omega)\} \) was increasing (or decreasing) by \( \pi \) every \( \omega_1 \) radians as well. So we not only to see a linear slope, but we also expect jumps upward every so often. The only graph that fits the bill is graph F.

The value \( c \) corresponds to the slope, which should be \(-\frac{t_1 + t_2}{2}\), as seen above.
3. Related Transforms

Part 1
\[ f_2[n] = -f[-n] \quad \Rightarrow \quad F_2(\Omega) = -F(-\Omega) \]

Part 2
\[ f_3[n] = f[n] - \delta[n] \quad \Rightarrow \quad F_3(\Omega) = F(\Omega) - 1 \]

Part 3
\[ f_4[n] = \text{Asym} \{ f[n] \} = \frac{f[n] - f[-n]}{2} \quad \Rightarrow \quad F_4(\Omega) = j \text{Im} \{ F(\Omega) \} \]

Part 4
\[ f_5[n] = f[n + 2] \quad \Rightarrow \quad F_5(\Omega) = e^{j2\Omega} F(\Omega) \]

Part 5
If we let:
\[ f_s[n] = \begin{cases} f[n/3] & \text{if } n \text{ is divisible by 3} \\ 0 & \text{otherwise} \end{cases} \]
Then we have \( f_6[n] = f_s[n] + f_s[n - 1] + f_s[n + 1] \), so:
\[ F_6(\Omega) = F_s(\Omega) \left( 1 + e^{j\Omega} + e^{-j\Omega} \right) = F_s(\Omega) (1 + 2 \cos(\Omega)) \]
Since \( F_s(\Omega) = F(3\Omega) \), we have:
\[ F_6(\Omega) = (F(3\Omega)) (1 + 2 \cos(\Omega)) \]
4. DFT

Part 1

We know that \( k = \frac{N}{f_s} \), so since \( k = \pm 268 \approx \pm \frac{f_s}{2} \), we know that we must have \( N \approx \frac{f_s}{2} \), so the only real possibility is \( N = 1024 \).

Part 2

Importantly, since the 'A' note (\( f = 1760 \text{Hz} \)) is above the Nyquist rate, it is going to alias down to \( f_{\text{apparent}} = f_s - 1760 \text{Hz} = 240 \text{Hz} \). Converting to \( k \), we find \( k = \pm 240 \left( \frac{1024}{2000} \right) \approx \pm 123 \).

We could also find this by first finding the expected \( k \) value for \( f = 1760 \text{Hz} \) first:
\[
k = \pm \left( \frac{1024}{2000} \right) 1760 \approx \pm 901.
\]
But that’s not in the range \([0, N/2]\), so it can’t be the final answer. Since the coefficients are periodic in \( N \), we need to find a value \( 1024m + 901 \), where \( m \) is an integer, for which that value falls in the range \([0, 512]\). That happens to be \( 1024 - 901 = 123 \).

Part 3

Since the 'C' note shows up at \( k = \pm 268 \) and the 'A' note (due to aliasing) shows up at \( k = \pm 123 \), the 'A' note must correspond to the peaks that are closer to 0, i.e., \( k_1 \).
5. DTFS Components

The first six of these problems can be solved directly:

Part 0

\[ X_0[k] = \frac{1}{6} \]

This signal has a constant magnitude of \( \frac{1}{6} \) (graph G), a constant real part of \( \frac{1}{6} \) (graph G), and a constant imaginary part of 0 (graph C).

Looking ahead, importantly, since all of \( x_0, x_1, x_2, x_3, x_4, \) and \( x_5 \) are just time-shifted versions of each other, they should all have the same magnitude (graph G).

Part 1

\[ X_1[k] = \frac{1}{6} e^{-j\frac{2\pi}{6}k} = \frac{1}{6} \left( \cos \left( \frac{2\pi}{6} k \right) - j \sin \left( \frac{2\pi}{6} k \right) \right) \]

The magnitude graph is still G, but now the real part looks like a cosine that goes through one period in \( K = 6 \) (graph I), and the imaginary part looks like an inverted sine that goes through one period in \( K = 6 \) (graph D).

Part 2

\[ X_2[k] = \frac{1}{6} e^{-j\frac{4\pi}{6}k} = \frac{1}{6} \left( \cos \left( \frac{4\pi}{6} k \right) - j \sin \left( \frac{4\pi}{6} k \right) \right) \]

This is the same, but now the cosine and sine each go through two full periods in \( K = 6 \), so magnitude is still G, real part is K, and imaginary part is E.

Part 3

This one is interesting, in that \( X_3[k] \) ends up being purely real:

\[ X_3[k] = \frac{1}{6} e^{-j\pi k} = \frac{1}{6} (-1)^k \]

Magnitude is G, real part is M, and imaginary part is C (no imaginary part).

Part 4

\[ X_4[k] = \frac{1}{6} e^{-j\frac{4\pi}{6}k} = \frac{1}{6} e^{j\frac{2\pi}{6}k} = \frac{1}{6} \left( \cos \left( \frac{4\pi}{6} k \right) + j \sin \left( \frac{4\pi}{6} k \right) \right) \]

This looks like \( X_2[k] \), but with the imaginary part negated, so magnitude is G, real part is K, and imaginary part is A.

Part 5

\[ X_5[k] = \frac{1}{6} e^{-j\frac{10\pi}{6}k} = \frac{1}{6} e^{j\frac{2\pi}{6}k} = \frac{1}{6} \left( \cos \left( \frac{4\pi}{6} k \right) + j \sin \left( \frac{4\pi}{6} k \right) \right) \]

This looks like \( X_1[k] \), but with the imaginary part negated, so magnitude is G, real part is I, and imaginary part is B.
The next 6 parts all have simple relationships to the first 6 parts. In particular, note that for all $i$, $y_i[n] = 1 - x_i[n]$. Thus, by linearity, we have $Y_i[k] = \delta[k] - X_i[k]$, which we can use to simplify our analysis.

Since all of the signals $y_i$ are just time-shifted versions of each other, all of the $Y_i$ will have the same magnitude.

And for each $Y_i[k]$, we will have $\text{Re}(Y_i[0]) = 1 - \text{Re}(X_i[0])$, and for all other $k$, $\text{Re}(Y_i[k]) = -\text{Re}(X_i[k])$.

Similarly, for all $k$, $\text{Im}(Y_i[k]) = -\text{Im}(X_i[k])$.

**Part 6**

$Y_0[k] = \delta[k] - X_0[k] = \delta[k] = 1/6$

Magnitude O, real part T, imaginary part C.

**Part 7**

$Y_1[k] = \delta[k] - X_1[k]$

Magnitude O, real part S, imaginary part B.

**Part 8**

$Y_2[k] = \delta[k] - X_2[k]$

Magnitude O, real part Q, imaginary part A.

**Part 9**

$Y_3[k] = \delta[k] - X_3[k]$

Magnitude O, real part N, imaginary part C.

**Part 10**

$Y_4[k] = \delta[k] - X_4[k]$

Magnitude O, real part Q, imaginary part E.

**Part 10**

$Y_5[k] = \delta[k] - X_5[k]$

Magnitude O, real part S, imaginary part D.