6.003

Practice Quiz 2: Solutions

1. Peaks and Valleys

Find the DTFT:

$$X(\Omega) = \frac{1}{1 - \alpha e^{-j\Omega m}}$$

The magnitude of the frequency response is

$$\begin{aligned}
\Big|H(\Omega)\Big| &= \frac{1}{\Big|1 - \alpha\cos(\Omega m) + j\alpha\sin(\Omega m)\Big|} \\
&= \frac{1}{\sqrt{(1-\alpha\cos(\Omega m))^2 + (\alpha\sin(\Omega m))^2}} \\
&= \frac{1}{\sqrt{1 - 2\alpha\cos(\Omega m) + \alpha^2}}
\end{aligned}$$

The maximum magnitude of $X(\omega)$ occurs when the magnitude of its denominator is at a minimum.

The minimum magnitude of the denominator is $\Big|1 - |\alpha|\Big|$.

Thus the maximum magnitude of $X(\Omega)$ is

$$\max = \left|\frac{1}{1 - |\alpha|}\right|$$

The minimum magnitude of $H(\Omega)$ occurs when the magnitude of its denominator is at a maximum.

The maximum magnitude of the denominator is $1 + |\alpha|$.

Thus the minimum magnitude of H is

$$\min = \frac{1}{1 + |\alpha|}$$

line	α	**max**	**min**	**matches**
1	0.7	3.3	0.6	H
2	−0.5	2	0.7	F
3	−0.8	5	0.6	A or E
4	−0.8	5	0.6	A or E

To determine which of A or E belongs on line 3, we can compute the DC magnitudes.

For line 3, the DC magnitude is $\frac{1}{1-(-0.8)}$ which is less than 1.

For line 4, the DC magnitude is $\frac{1}{1-0.8}$ which is 5.

Thus E belongs on line 4 and A belongs on line 3.

2. DFTs

$x_1[n]$ shows two full cycles of a triangle wave. Therefore the fundamental frequency of the triangle wave falls at $k = 2$. There could also be harmonics of $k = 2$ (i.e., at $k = 4, 6, 8, ...$).

$\rightarrow x_1$ matches plot E

Both $x_2[n]$ and $x_3[n]$ show 1.5 full cycles of a triangle wave. Therefore the magnitude will peak between $k = 1$ and $k = 2$. If $x_2[n]$ is periodically extended, it will have a big discontinuity between periods. $x_3[n]$ has a much smaller discontinuity. Also, the DC value of $x_3[n]$ is much larger than that of $x_2[n]$.

$\rightarrow x_2$ matches plot D, x_3 matches plot A

$x_4[n]$ shows 1 full cycle of a triangle wave. Therefore we should see a peak value at $k = 1$, and we should see nonzero values at odd values of k.

$\rightarrow x_4$ matches plot F

When periodically extended, $x_5[n]$ will be a sawtooth with $k = 1$. There will also be a large discontinuity at the period boundaries, so that will generate contributions smeared out across the whole spectrum.

$\rightarrow x_5$ matches plot B

When periodically extended, $x_6[n]$ will make a triangle wave at $k = 1$. Notice however that there is a large DC component.

$\rightarrow x_6$ matches plot C

3. Trigonometric Fourier Series

Part 1

We can use trig identities or Euler's formula to reduce the expression for $f(t)$ to the standard trigonometric Fourier series form.

$$\begin{aligned}
f(t) &= 2\cos\left(\frac{1}{3}\pi t\right)\sin\left(\frac{1}{4}\pi t\right) + 3 \\
&= 2\frac{1}{2}(e^{j\pi t/3} + e^{-j\pi t/3})\frac{1}{2j}(e^{j\pi t/4} - e^{-j\pi t/4}) + 3 \\
&= \frac{1}{2j}\left(e^{j7\pi t/12} - e^{-j7\pi t/12} - e^{j\pi t/12} + e^{-j\pi t/12}\right) + 3 \\
&= \sin\left(\frac{7\pi t}{12}\right) - \sin\left(\frac{\pi t}{12}\right) + 3 \\
&= \sin\left(\frac{2\pi t}{24}7\right) - \sin\left(\frac{2\pi t}{24}1\right) + 3
\end{aligned}$$

The result has the desired form if $T = 24$ and $M = 7$. There are three non-zero terms:

$c_0 = 3$
$d_1 = -1$
$d_7 = 1$

Part 2

Using the plots and the fact that $F[k]$ is periodic in k with period $N = 5$, we can see that

$F[0] = 1$
$F[1] = -F[-1] = j2$
$F[2] = F[-2] = -3$

It follows that we can write $f[n]$ as

$$f[n] = 1 - 4\sin\left(\frac{2\pi}{N}n\right) - 6\cos\left(\frac{4\pi}{N}n\right)$$

Therefore, $M = 2$ and

$c_0 = 1$
$d_1 = -4$
$c_2 = -6$

4. Signal Facts

Fact 1 tells us that our Fourier series must be conjugate symmetric, i.e., that $X[k] = X^*[-k]$ for all k.

Fact 3 tells us that the $x[n]$ is periodic in $N = 6$ (though this alone does not tell us that $N = 6$ is the fundamental period of the signal).

Fact 4 tells us that $X[2] = 0$, which, by the conjugate symmetry, tells us that $X[-2] = X[4]$ is also 0.

In order for fact 5 to hold, $X[0]$ must be 1 ($y[n]$ being antisymmetric means that *its* DC component must be 0, since $x[n] = y[n] + 1$, $x[\cdot]$'s DC component must be 1.

Fact 6 tells us that $X[3] = 0$, since:

$$\frac{1}{6}\sum_{n=-2}^{3}(-1)^n x[n] = \frac{1}{6}\sum_{n=-2}^{3}(e^{-j\pi n})x[n] = \frac{1}{6}\sum_{n=-2}^{3}(e^{-j\frac{2\pi 3}{6}n})x[n] = X[3]$$

The only remaining pieces to figure out are $X[1]$ and $X[-1] = X[5]$. Since the function with its DC component removed is purely antisymmetric, we know that $X[1]$ and $X[5]$ must be purely imaginary. Combining this with fact 7, we find that $X[1] = mj$ and $X[5] = -mj$ for some value of m.

Putting all these together, we find that $x[n] = 1 - 2m\sin(\frac{\pi}{3}n)$.

Fact 2 tells us that the maximum value $x[\cdot]$ reaches is 5. Thus, we need m such that $2m \times \max_n \left(\sin\left(\frac{\pi}{3}n\right)\right) = 5$

The values that $\sin\left(\frac{\pi}{3}n\right)$ takes in one period are:

$\sin(0) = 0$ $\quad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ $\quad \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$ $\quad \sin(\pi) = 0$

$\sin\left(\frac{4\pi}{3}\right) = -\sin\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ $\quad \sin\left(\frac{5\pi}{3}\right) = -\sin\left(\frac{1\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

Using this and solving, we find that $m = \frac{4}{\sqrt{3}}$, which gives:

$$x[n] = 1 - \frac{8}{\sqrt{3}}\sin\left(\frac{\pi}{3}n\right)$$

5. DTFT Matching

We can start by noticing some properties of $x_1[n]$, and noting their effects on the Fourier transform:

- $x_1[n]$ is a real-valued, symmetric function of n. This tells us that the Fourier transform must also be a real-valued, symmetric function of Ω.
- $x_1[n]$ is periodic (in N_0). This tells us that it could be represented by a Fourier series, and thus the Fourier transform will only have nonzero values at discrete frequencies (specifically, at multiples of $\frac{2\pi}{N_0}$).
- $x_2[n]$ is a real-valued eternal cosine. Its Fourier transform must consist of two delta functions, real-valued and symmetric about $\Omega = 0$.

Just using these result, we can eliminate all of the graphs except 5 and 7. But in order to differentiate between those two, and to find the values of A, N_0, B, and Ω_0, we need to do a bit more work.

We can start by finding $X_1(\Omega)$, which we can do by using the Fourier *series* as a starting point. Note that if $X_1[k]$ is the DTFS of x_1 (analyzed with $N = N_0$), then:

$$X_1(\Omega) = 2\pi \sum_{k=0}^{N_0-1} X_1[k]\delta\left(\Omega - \frac{2\pi k}{N_0}\right)$$

i.e., the DTFT is only nonzero at the frequencies represented in the DTFS. But because reconstructing $x_1[n]$ requires integrating, we need to have nonzero energy there (hence the delta functions). With those delta functions, we get the same thing from our inverse DTFT that we would get from our inverse DTFS (a sum of complex exponentials only at integer multiples of the fundamental frequency, each scaled by the proper amount).

Even before we solve for $X_1[k]$, we can find N and Ω_0. Since the harmonically-related frequencies present in both graph 5 and graph 7 are integer multiples of $\pi/4$, we can find that $N = 8$. Then, the one frequency that isn't a part of that harmonic sequence must be associated with $x_2[n]$, and so $\Omega_0 = \pi/3$.

We can also solve for B by looking more closely at the height of the delta functions at $\pm\pi/3$. Applying the synthesis equation just to those two points, we find that $B = \frac{2}{\pi}$:

$$\begin{aligned}
B\cos\left(\frac{\pi}{3}n\right) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(2\delta\left(\Omega - \frac{\pi}{3}\right) + 2\delta\left(\Omega - \frac{\pi}{3}\right)\right)e^{j\Omega n}\mathrm{d}\Omega \\
&= \frac{1}{\pi}\left(e^{j\frac{\pi}{3}n} + e^{-j\frac{\pi}{3n}}\right) \\
&= \frac{2}{\pi}\cos\left(\frac{\pi}{3}n\right)
\end{aligned}$$

At this point, we still need to solve for A, and to determine the shape of the graph. As suggested above, we can do this by finding the Fourier *series* $X_1[k]$, from which we can find the Fourier transform.

$$X_1[k] = \frac{1}{N}\sum_{n=\langle N\rangle} x_1[n]e^{-j\frac{2\pi k}{N}n} = \frac{A}{N} + \frac{2A}{N}\cos\left(\frac{2\pi k}{N}\right)$$

That this is shaped like a cosine tells us that graph 5 is the correct graph (rather than graph 7).

And finally, we can find A by looking at $X_1(0)$.

From plugging in to our result from above, we know that:

$X_1(0) = 2\pi X_1[0]\delta(0) = 2\pi \times \frac{3A}{N}\delta(0) = \frac{2\pi\times 3A}{8}\delta(0)$

Matching with the graph, we know that this expression must also be equal to $X(0) = 2\delta(0)$, so:

$$\frac{6\pi A}{8} = 2 \qquad \Rightarrow \qquad A = \frac{8}{3\pi}$$