6.003

Practice Quiz 1: Solutions

1. Fourier Transforms

Part 1

The signal $x(t)$ can be written as the difference between a rectangular pulse that extends from -3 to 3 and a rectangular pulse that extends from -2 to 2. The Fourier transform of the former is

$$\int_{-3}^{3} e^{-j\omega t}dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-3}^{3} = 2\frac{\sin(3\omega)}{\omega}$$

Similarly, the Fourier transform of the latter is $2\frac{\sin(2\omega)}{\omega}$. Thus the total answer is

$$X(\omega) = 2\frac{\sin(3\omega)}{\omega} - 2\frac{\sin(2\omega)}{\omega}$$

Part 2

Calculate the Fourier transform of the right half of this function:

$$X_r(\omega) = \int_0^1 e^t e^{-j\omega t}dt = \int_0^1 e^{(1-j\omega)t}dt = \left.\frac{e^{(1-j\omega)t}}{(1-j\omega)}\right|_0^1 = \frac{e^{(1-j\omega)}-1}{1-j\omega}$$

The Fourier transform of the left part is

$$X_l(\omega) = X_r(-\omega) = \frac{e^{(1+j\omega)}-1}{1+j\omega}$$

Therefore

$$X(\omega) = X_r(\omega) + X_l(\omega) = \frac{e^{(1-j\omega)}-1}{1-j\omega} + \frac{e^{(1+j\omega)}-1}{1+j\omega} = \frac{2e\cos(\omega) + 2e\omega\sin(\omega) - 2}{1+\omega^2}$$

Part 3

Start by splitting the signal into two pieces:

$x_3[n] = x_{3a}[n] + x_{3b}[n]$, where:

$$x_{3a}[n] = \begin{cases} (1/2)^{n/2} & \text{if } n \geq 0 \text{ and } n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

and

$$x_{3b}[n] = \begin{cases} (1/3)^{(n+1)/2} & \text{if } n > 0 \text{ and } n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

By linearity, we can compute $X_{3a}(\cdot)$ and $X_{3b}(\cdot)$ and add them together to find $X(\Omega)$. We start with $X_{3a}(\cdot)$:

$$X_{3a}(\Omega) = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \left(\frac{1}{2}\right)^{n/2} e^{-j\Omega n}$$

And substitute $m = n/2$

$$X_{3a}(\Omega) = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \left(\frac{1}{2}\right)^{n/2} e^{-j\Omega n} = \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{m} e^{-j2\Omega m} = \sum_{m=0}^{\infty} \left(\frac{1}{2}e^{-j2\Omega}\right)^{m} = \frac{1}{1-\frac{1}{2}e^{-j2\Omega}}$$

We can use our analysis from above to making thinking about $X_{3b}(\cdot)$ easier by defining another signal $x_{3c}[\cdot]$ such that $x_{3b}[n] = \dfrac{x_{3c}[n-1]}{3}$. If we do this, then we can find $X_{3c}(\cdot)$ using almost the same analysis as above:

$$X_{3c}(\Omega) = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \left(\frac{1}{3}\right)^{n/2} e^{-j\Omega n} = \sum_{m=0}^{\infty} \left(\frac{1}{3}\right)^{m} e^{-j2\Omega m} = \sum_{m=0}^{\infty} \left(\frac{1}{3}e^{-j2\Omega}\right)^{m} = \frac{1}{1-\frac{1}{3}e^{-j2\Omega}}$$

Then, using the linearity and time shift properties, we can find $X_{3b}(\cdot)$:

$$X_{3b}(\Omega) = \left(\frac{1}{3}e^{-j\Omega}\right)\left(\frac{1}{1-\frac{1}{3}e^{-j2\Omega}}\right)$$

Finally, by linearity, we have:

$$X_3(\Omega) = X_{3a}(\Omega) + X_{3b}(\Omega) = \frac{1}{1-\frac{1}{2}e^{-j2\Omega}} + \frac{\frac{1}{3}e^{-j\Omega}}{1-\frac{1}{3}e^{-j2\Omega}}$$

2. Find the Magnitude

Part 1.

$$X_1[k] = \sum_{n=0}^{19} x_1[n] e^{-j\frac{2\pi k}{20}n} = 1 - e^{-j\frac{2\pi k}{20}2} = e^{-j\frac{2\pi k}{20}}\left(e^{j\frac{2\pi k}{20}} - e^{-j\frac{2\pi k}{20}}\right) = e^{-j\frac{2\pi k}{20}} j2\sin\left(\frac{2\pi k}{20}\right)$$

$$\left|X_1[k]\right| = 2\left|\sin\left(\frac{2\pi k}{20}\right)\right|$$

Notice that $\left|X_1[k]\right| = 0$ when $k = 0, 10, 20, \ldots \rightarrow$ Panel D.

Part 2.

$$X_2[k] = \sum_{n=0}^{19} x_2[n] e^{-j\frac{2\pi k}{20}n} = 1 + e^{-j\frac{2\pi k}{20}3} = e^{-j\frac{2\pi k}{40}3}\left(e^{j\frac{2\pi k}{40}3} + e^{-j\frac{2\pi k}{40}3}\right) = e^{-j\frac{2\pi k}{40}3} 2\cos\left(\frac{2\pi k}{40}3\right)$$

$$\left|X_2[k]\right| = 2\left|\cos\left(\frac{2\pi k}{40}3\right)\right|$$

Notice that the cosine function would be 1 if $k = 0, \frac{20}{3}, \frac{40}{3}, 20, \ldots$. Since k must be an integer,the cosine function is near a peak at $k = 0, 7, 13, 20, \ldots$ and it is near a minimum between these values. Therefore the answer is Panel B.

Part 3.

$$X_3[k] = \sum_{n=0}^{19} x_3[n] e^{-j\frac{2\pi k}{20}n} = 1 + e^{-j\frac{2\pi k}{20}2} = e^{-j\frac{2\pi k}{20}}\left(e^{j\frac{2\pi k}{20}} + e^{-j\frac{2\pi k}{20}}\right) = e^{-j\frac{2\pi k}{20}} 2\cos\left(\frac{2\pi k}{20}\right)$$

$$\left|X_3[k]\right| = 2\left|\cos\left(\frac{2\pi k}{20}\right)\right|$$

Notice that $\left|X_3[k]\right| = 1$ when $k = 0, 10, 20, \ldots$ and is at a minimum between these values. Thus the answer is Panel A.

Part 4.

$$X_4[k] = \sum_{n=0}^{19} x_4[n] e^{-j\frac{2\pi k}{20}n} = 1 - e^{-j\frac{2\pi k}{20}10} = e^{-j\frac{2\pi k}{20}5}\left(e^{j\frac{2\pi k}{20}5} - e^{-j\frac{2\pi k}{20}5}\right) = e^{-j\frac{2\pi k}{20}} j2\sin\left(\frac{2\pi k}{20}5\right)$$

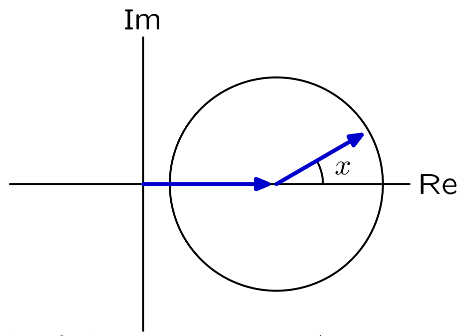
$$\left|X_4[k]\right| = 2\left|\sin\left(\frac{2\pi k}{20}5\right)\right|$$

Notice that $\left|X_4[k]\right| = 0$ when $k = 0, 2, 4, 6, 8, \ldots$. This pattern doesn't match any of the panels. Therefore the answer is **None**.

3. Angular Trends

- $\angle e^{-jx}$: A complex exponential of the form $e^{j\theta}$ has magnitude 1 and angle θ. Therefore, the angle of e^{-jx} is $-x$, as shown in plot B.

- $\angle(1+0.8e^{jx})$: The number $1+0.8e^{jx}$ is the sum of 1 with a vector of magnitude 0.8 and angle x as shown in the following plot.



  When x is small, the angle of the sum is zero. As x increases, the angle increases until x reaches about $3\pi/4$. At this point, the angle of the sum is on the order of $\pi/3$. As x increases above $3\pi/4$, the angle of the sum quickly decreases, returning to zero when $x = \pi$. From the symmetry of the figure, it follows that the angle of the sum is an odd function of x. Thus the answer is plot E.

- $\angle\left(\frac{1+0.4e^{jx}}{2+0.8e^{jx}}\right)$: Since the denominator is twice the numerator, this is just the angle of a real number (1/2), which is zero – plot I.

- $\angle(1+e^{jx})$:

$$1+e^{jx} = e^{-j\frac{x}{2}}\left(e^{j\frac{x}{2}}+e^{-j\frac{x}{2}}\right) = e^{-j\frac{x}{2}}\,2\cos\left(\frac{x}{2}\right)$$

  Thus the angle of $1+e^{jx}$ is $x/2$ for $-\pi < x < \pi$. At $x = \pi$ the sign of the cosine flips so that angle jumps by π. Thus the answer is plot C.

- $\angle(1+0.8e^{j2x})$: This expression looks like part 2 (above) except x is replaced by $2x$. Therefore the answer the same as that for part 2 except that the x-axis is compressed by a factor of 2 – generating plot G.

- $\angle(0.9e^{jx}+0.8e^{-jx})$: The expression $0.9e^{jx}+0.8e^{-jx}$ can be simplified by converting to Cartesian form:

$$0.9\cos(x)+j0.9\sin(x)+0.8\cos(x)-j0.8\sin(x) = 1.7\cos(x)+j0.1\sin(x)$$

  The angle is therefore $\arctan\left(\frac{0.1\sin(x)}{1.7\cos(x)}\right) = \arctan\left(\frac{1}{17}\tan(x)\right)$ which is plot J.

- $\angle\left(\frac{1}{1+0.8e^{jx}}\right)$: The expression $1+0.8e^{jx}$ was evaluated in part 2 (above). Here the expression is in the denominator, so the answer is the negative of the answer to part 2 – which yields plot F.

4. More than Meets the Eye

Part 1

We can start by defining smaller signals to try to make our analysis easier (so that we can leverage properties rather than doing the direct integral). Here, we'll define the following signals:

- $x_{1a}(t) = e^{-t}u(t)$
- $x_{1b}(t) = x_{1a}(2t) = e^{-2t}u(t)$
- $x_{1c}(t) = tx_{1b}(t) = te^{-2t}u(t)$
- $x_{1d}(t) = x_{1c}(t) - x_{1c}(-t) = te^{-2|t|}$
- $x_1(t) = 3x_{1d}(t) = 3te^{-2|t|}$

The main idea here is that we can compute one relatively straightforward Fourier transform the long way, and then let properties carry us the rest of the way to our solution for the original problem.

So by hand, we compute:

$$X_{1a}(\omega) = \int_0^\infty e^{-t}e^{-j\omega t}\mathrm{d}t = \frac{1}{1+j\omega}$$

Then, by the time scaling property, we have:

$$X_{1b}(\omega) = \frac{1}{2}X_{1a}\left(\frac{\omega}{2}\right) = \frac{1}{2+j\omega}$$

By the frequency derivative property, we have:

$$X_{1c}(\omega) = j\frac{\mathrm{d}}{\mathrm{d}\omega}X_{1b}(\omega) = \frac{1}{(2+j\omega)^2}$$

By linearity and the time reversal property, we have:

$$X_{1d}(\omega) = X_{1c}(\omega) - X_{1c}(-\omega) = \frac{1}{(2+j\omega)^2} - \frac{1}{(2-j\omega)^2}$$

And, finally, by linearity:

$$X_1(\omega) = 3X_{1d}(\omega) = \frac{1}{(2+j\omega)^2} - \frac{1}{(2-j\omega)^2} = \frac{24j\omega}{(4+\omega^2)^2}$$

Part 2

Duality: If $x(t) \overset{\text{FT}}{\Longleftrightarrow} X(\omega)$ then

$$X(t) \overset{\text{FT}}{\Longleftrightarrow} 2\pi x(-\omega)$$

From part 1,

$$3te^{-2|t|} \overset{\text{FT}}{\Longleftrightarrow} -\frac{24j\omega}{(4+\omega^2)^2}$$

So by duality, if

$$-\frac{24jt}{(4+t^2)^2} \overset{\text{FT}}{\Longleftrightarrow} 2\pi(-3\omega e^{-2|\omega|})$$

then

$$\frac{j12t}{\pi(4+t^2)^2} \overset{\text{FT}}{\Longleftrightarrow} 3\omega e^{-2|\omega|}$$

Part 3

Let

$$z(t) = \dot{x}(t) = \frac{dx(t)}{dt}\,.$$

Then

$$Z(\omega) = j\omega X(\omega)\,.$$

We can express $y(t)$ in terms of z as

$$y(t) = z\big(3(t+5)\big)$$

and then

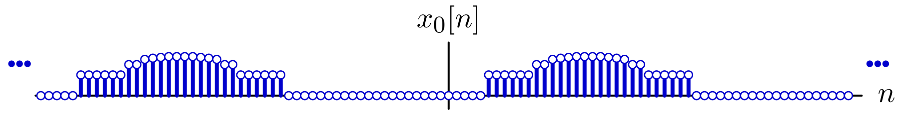
$$\begin{aligned} Y(\omega) &= \int y(t)e^{-j\omega t}dt \\ &= \int z\big(3(t+5)\big)e^{-j\omega t}dt \end{aligned}$$

Let $\tau = 3(t+5)$ then $d\tau = 3dt$ and

$$\begin{aligned} Y(\omega) &= \int z(\tau)e^{-j\omega\left(\frac{\tau}{3}-5\right)}\frac{1}{3}d\tau \\ &= \frac{1}{3}e^{j5\omega}\int z(\tau)e^{-j\frac{\omega}{3}\tau}d\tau \\ &= \frac{1}{3}e^{j5\omega}Z\left(\frac{\omega}{3}\right) \\ &= \frac{1}{3}e^{j5\omega}j\frac{\omega}{3}X\left(\frac{\omega}{3}\right) \end{aligned}$$

5. Dome, Sweet Dome

Note that the original signal is periodic in $N = 51$ as shown below.



Answers and rationale for all parts follow:

- X_A

$$X_A[k] = \mathrm{Re}\left(X_0[k]\right) = \frac{1}{2}X_0[k] + \frac{1}{2}X_0^*[k] \quad \text{(property of complex numbers)}$$

Now find the effect of conjugating $X[k]$.

$$X[k] = \frac{1}{N}\sum x[n]e^{-\frac{j2\pi kn}{N}} \quad \text{(Fourier analysis equation)}$$

$$X^*[k] = \frac{1}{N}\sum x^*[n]e^{\frac{j2\pi kn}{N}} \quad \text{(conjugate both sides)}$$

$$X^*[k] = \frac{1}{N}\sum x^*[-n]e^{-\frac{j2\pi kn}{N}} \quad (n \to -n)$$

$$x^*[-n] \overset{\text{FT}}{\Longleftrightarrow} X^*[k] \quad \text{(Fourier analysis equation)}$$

Then

$$x_A[n] = \frac{1}{2}x_0[n] + \frac{1}{2}x_0^*[-n] = \frac{1}{2}x_0[n] + \frac{1}{2}x_0[-n]$$

since $x_0[n]$ is real-valued. The flipped signal $x_0[-n]$ looks a lot like $x_0[n]$ (since that function is symmetric about $n = 18.5$) but it is shifted by 15 samples. Thus when $x_0[n]$ is added to $x_0[-n]$, part of the dome from $x_0[n]$ overlaps part of the dome from $x_0[-n]$. The result looks like $x_{16}[n]$.

We can think about symmetry properties as a way to check this answer. The sum of $x_0[n]$ and $x_0[-n]$ (which is a flipped version about $n = 0$) will be an even function of n. Since $x_0[n]$ is also periodic in $n = 51$, the result of adding $x_0[n]$ to $x_0[-n]$ is also symmetric about $n = 25.5$. There are only four signals with this symmetry: x_9, x_{11}, x_{16}, and x_{22}. (Notice that x_{14} is not quite right since there are only four leading values of zero.) However, the signal is clearly not zero, eliminating x_{11}. Also x_9 is upside down and x_{22} is upside-down plus a constant. Thus the answer must be x_{16}.

- X_B

$$X_B[k] = \text{Im}\left(X_0[k]\right) = \frac{1}{2j}X_0[k] - \frac{1}{2j}X_0^*[k] \quad \text{(property of complex numbers)}$$

Then

$$x_B[n] = \frac{1}{2j}x_0[n] - \frac{1}{2j}x_0^*[-n] = \frac{1}{2j}x_0[n] - \frac{1}{2j}x_0[-n]$$

Since $x_0[n]$ is real-valued, $x_B[n]$ must be complex-valued.

None of the possible answers are complex valued, so the answer is "COMPLEX".

- X_C

$$X_C[k] = j\,\text{Im}\left(X_0[k]\right) = \frac{1}{2}X_0[k] - \frac{1}{2}X_0^*[k] \quad \text{(property of complex numbers)}$$

Thus

$$x_C[n] = \frac{1}{2}x_0[n] - \frac{1}{2}x_0^*[-n] = \frac{1}{2}x_0[n] - \frac{1}{2}x_0[-n]$$

since $x_0[n]$ is real-valued. When $x_0[-n]$ is subtracted from $x_0[n]$, the result is an odd function of n. Since $x_0[n]$ is also periodic in $N = 51$, the result is also antisymmetric about $n = 25.5$. The result looks like $x_8[n]$.

x_{11} has the right symmetry properties, but our answer is clearly not zero. Also x_{13} clearly has the wrong shape. x_{21} is the negative of the right answer, i.e.,$x[-n] - x[n]$. So the answer must be x_8.

- X_D

By setting $k = 0$ in the analysis equation,

$$X_0[k] = \frac{1}{N}\sum x_0[n]e^{\frac{-j2\pi kn}{N}}$$

we can see $X_0[0]$ is the average value of $x_0[n]$. Let $\bar{x}$ represent the average value of $x_0[n]$. Then by linearity

$$x_0[n] - \bar{x} \overset{\text{FT}}{\Longleftrightarrow} X_0[k] - X_0[0]$$

Setting $X_0[0]$ to zero is thus equivalent to subtracting the average value of $x_0[n]$ from $x[n]$ for all n.

Two signals $x_6[n]$ and $x_{19}[n]$ are simple vertical shifts of $x_0[n]$. Since $x_{19}[n]$ is shifted in the wrong direction, the answer must be $x_6[n]$.

- X_E

  Setting the twenty-fifth component of the Fourier series to zero is equivalent to subtracting a complex exponential with frequency of $\frac{2\pi 25}{51}$ from $x_0[n]$.

  So our new signal would be $x_E[n] = x_0[n] - X_0[25]e^{j2\pi(25/51)n}$. Unless $X_0[25] = 0$, this extra term will be complex-valued.

- X_F

  By linearity, adding a constant to $X_0[k]$ adds a signal $y[n]$ to $x_0[n]$ where $y[n]$ is the signal whose Fourier series $Y[k]$ is $1/51$ for all k:

  $$y[n] = \sum \frac{1}{51} e^{\frac{j\omega kn}{51}}$$

  By orthogonality, $y[n]$ must be $\delta[n]$ since the above sum goes to zero except at $n = 0$.

  Thus the solution is $x_{20}[n]$.

- X_G

  The multiplier $e^{j\pi}$ is equal to -1. Therefore the new signal is flipped about the horizontal axis. The solution must be $x_{23}[n]$.

- X_H

  The multiplier here is the same as in Part 7.

  However, the DC term is still that of the original signal (which is positive). The resulting effect is that $x_H[n] = 2X_0[0] - x_0[n]$ (i.e., it is reflected about the horizontal axis, and then shifted to account for the change in DC value).

  The solution is $x_{15}[n]$.

- X_I

  Negating the angle of a complex number while holding the magnitude constant has the same effect as taking the complex conjugate of the original number. This follows from thinking about the definition of magnitude and angle of a complex number a:

  $$a = |a|e^{j\angle a}$$
  $$a^* = |a|e^{-j\angle a}$$

  Thus $X_I[k] = X_0^*[k]$.

  Conjugating the Fourier series has the effect of conjugating the time function and then flipping it about $n = 0$. Since $x_0[n]$ is real-valued, the result is just a time flip, and the answer is x_{10}.