6.003: Signal Processing

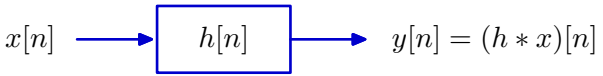
Filtering with DFT

8 April 2021

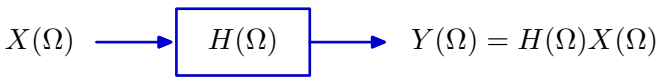
Filtering

We can view filtering in both the time and frequency domains:

Time Domain:



Frequency Domain:



Each frequency component $X(\Omega)$ is scaled by a factor $H(\Omega)$, which can be possibly complex.

The system is completely described by the set of scale factors $H(\cdot)$, which we refer to as the **frequency response** of the system.

Filtering

In lecture 8B: constructed a time-domain filter (described by its unit sample response $h[\cdot]$) by reasoning about its frequency response $H(\cdot)$.

Our focus was on creating a filter that boosted low frequency content in an input signal. We started by first creating a *low-pass* filter that removes high frequencies.

To this end, we started with a filter defined by $H(\Omega) = \frac{1}{2} + \frac{1}{2}\cos(\Omega)$, which has the nice property that it goes to 0 around $\Omega = \pm\pi$ and it goes to 1 at $\Omega = 0$ (i.e., the highest frequencies are the most attenuated).

Let's explore this filter a little bit, and see what happens when we apply it to a piece of music.

Check Yourself!

If $H(\Omega) = \frac{1}{2} + \frac{1}{2}\cos(\Omega)$, what is the shape of $h[n]$?
For what values of n is $h[n]$ nonzero?

Check Yourself!

If $H_2(\Omega) = (H(\Omega))^2 = \left(\frac{1}{2} + \frac{1}{2}\cos(\Omega)\right)^2$, what is the shape of $h_2[n]$? For what values of n is $h_2[n]$ nonzero?

Check Yourself!

How about $H_L(\Omega) = (H(\Omega))^{1000} = \left(\frac{1}{2} + \frac{1}{2}\cos(\Omega)\right)^{1000}$?

What does $h_L[n]$ look like? How many nonzero values do we expect?

Check Yourself!

$h_L[\cdot]$ was a decent low-pass filter, but our original goal was bass boost.

Let $x[\cdot]$ be our original input, and $(x * h_L)[\cdot]$ represent a low-passed version of $x[\cdot]$.

We want to use a single convolution to produce a new signal with the high frequencies still present, but the low frequencies amplified:

$$(x * h_B)[n] = x[n] + c(x * h_L)[n]$$

Find an expression for $h_B[n]$.

Downsides of This Approach

The approach we took last time was to determine a desired frequency response $H(\cdot)$, compute the unit sample response $h[\cdot]$ of that filter, and then convolve an input signal with $h[\cdot]$.

This approach worked reasonably well, but it has a few downsides:

- Required a frequency response for which $h[\cdot]$ is reasonably straightforward to calculate.
- Long unit sample responses $h[\cdot]$ can mean that the convolution itself can take a long time.

There are many ways to overcome these issues (some of which are beyond the scope of 6.003). We'll take a look at one such approach today.

Filtering in the Frequency Domain

Could we have avoided these issues by working exclusively in the frequency domain?

Yes, kind of. We certainly could have multiplied $X(\cdot)$ by some desired frequency response $H(\cdot)$ and then converted back:

$$y[n] = (x * h)[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) H(\Omega) e^{j\Omega n} d\Omega$$

But this approach is not super great, either. Why?

An Alternative Approach: Use the DFT

If we know the frequency response we're interested in, we can take a different approach:

- compute the DFT of $x[\cdot]$
- scale each frequency component by the appropriate $H(\Omega)$ value
- compute the inverse DFT

For our low-pass filter, this involves scaling certain frequency components by some positive value, and leaving the others unchanged.

Check Yourself!

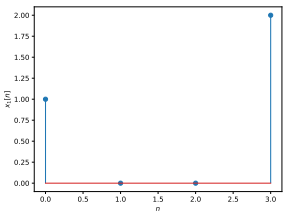
What are all of the issues with the following code for applying such a filter?

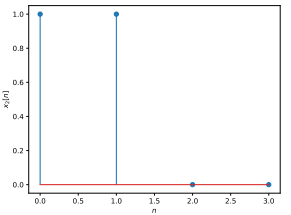
```
from lib6003.fft import fft, ifft
from lib6003.audio import wav_read, wav_write

x, fs = wav_read('super.wav')
N = len(x)

fc = 400 # cutoff frequency in Hz; boost all frequency content below fc

X = fft(x)
Y = X[:]  # copy values from X
Y[:fc] = [i*5 for i in X[:fc]]

y = ifft(Y)
assert all(abs(i.imag) < 1e-12 for i in y)
wav_write([i.real for i in y], fs, 'super_bb.wav')
```

Filtering with DFT

What if we want to apply a filter for which we don't know the frequency response $H(\Omega)$, but we *do* know the unit sample response $h[n]$?

Consider two simpler signals, shown below (and assumed to be zero outside the indicated range):



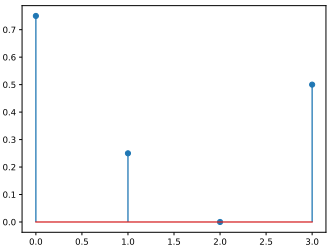


What does $(x_1 * x_2)[n]$ look like?
What does the following signal look like?

```
y = ifft([i*j for i,j in zip(fft(x1), fft(x2))])
```

Filtering with DFT

```
from lib6003.fft import *
import matplotlib.pyplot as plt

x1 = [1, 0, 0, 2]
x2 = [1, 1, 0, 0]

y = ifft([i*j for i,j in zip(fft(x1), fft(x2))])
plt.stem([i.real for i in y])
plt.show()
```



:O :(

Filtering with DFT

Let's look more closely to see why we get this surprising result!

$$x_1[n] = \delta[n] + 2\delta[n-3]$$
$$x_2[n] = \delta[n] + \delta[n-1]$$

Let $y = (x_1 * x_2)[n]$, then $Y(\Omega) = X_1(\Omega)X_2(\Omega)$

$$X_1(\Omega) = 1 + 2e^{-j3\Omega}$$
$$X_2(\Omega) = 1 + e^{-j\Omega}$$
$$Y(\Omega) = 1 + e^{-j\Omega} + 2e^{-j3\Omega} + 2e^{-j4\Omega}$$

Double checking, we find that

$$y[n] = \delta[n] + \delta[n-1] + 2\delta[n-3] + 2\delta[n-4] \quad \checkmark$$

Check Yourself!

Let's now think about $Y'[k] = \frac{1}{N} Y\left(\frac{2\pi k}{N}\right)$. With $N = 4$, we have:

$$\begin{aligned} Y'[k] &= \frac{1}{4} Y\left(\frac{\pi}{2} k\right) \\ &= \frac{1}{4}\left(1 + e^{-j\frac{\pi}{2}k} + 2e^{-j\frac{3\pi}{2}k} + 2e^{-j\frac{4\pi}{2}k}\right) \end{aligned}$$

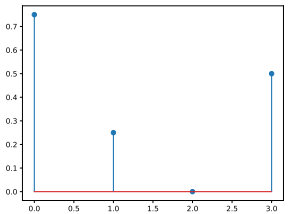
If we compute the inverse DFT to find $y[n]$, what do we find?

Extra Scaling Factor

What about the extra scaling factor? We have

$y[n] = y[n+4] = 3\delta[n] + \delta[n-1] + 2\delta[n-3]$

But that's not quite what we saw in our graph from earlier:



It has the right shape, but differs in scale.

Extra Scaling Factor

Remember that $Y'[k] = \frac{1}{4}\left(1 + e^{-j\frac{\pi}{2}k} + 2e^{-j\frac{3\pi}{2}k} + 2e^{-j\frac{4\pi}{2}k}\right)$

We computed the values in that plot by multiplying the DFT of $x_1[\cdot]$ by the DFT of $x_2[\cdot]$. Let's see where the math takes us if we do out that analysis:

Generalizing

Comparing multiplication of DFTs to multiplication of DTFTs, we found that multiplying DFTs:

- led to aliasing of some samples in the convolution formula (if $(x * h)[n]$ was nonzero for $n \geq N$ or $n < 0$, those values will alias back to values in the range $0 \leq n < N$).
- led to an additional scaling factor of $\frac{1}{N}$

We'll formalize these two ideas on the following slide

Circular Convolution

Multiplication of DFTs corresponds to **circular** convolution (and scaling by $1/N$) in time. Assume that $F[k]$ is the product of the DFTs of $f_1[n]$ and $f_2[n]$.

$$
\begin{aligned}
f[n] &= \sum_{k=0}^{N-1} F[k] e^{j\frac{2\pi k}{N}n} = \sum_{k=0}^{N-1} F_1[k]F_2[k] e^{j\frac{2\pi k}{N}n} \\
&= \sum_{k=0}^{N-1} F_1[k] \Big(\frac{1}{N} \sum_{m=0}^{N-1} f_2[m] e^{-j\frac{2\pi k}{N}m} \Big) e^{j\frac{2\pi k}{N}n} \\
&= \frac{1}{N} \sum_{m=0}^{N-1} f_2[m] \sum_{k=0}^{N-1} F_1[k] e^{j\frac{2\pi k}{N}(n-m)} \\
&= \frac{1}{N} \sum_{m=0}^{N-1} f_2[m] f_{1p}[n-m]
\end{aligned}
$$

where $f_{1p}[n] = f_1[n \bmod N]$ is a periodically extended version of $f_1[n]$.

We refer to this as **circular** or **periodic** convolution:

$$\frac{1}{N}(f_1 \circledast f_2)[n] \quad \overset{\text{DFT}}{\Longleftrightarrow} \quad F_1[k] \times F_2[k]$$

Circular Convolution

$$X_1[k] \times X_2[k] \overset{\text{DFT}}{\Longleftrightarrow} \frac{1}{N}(x_1 \circledast x_2)[n]$$

where $\circledast$ denotes the **circular convolution** operator.

The result of circular convolution is equivalent to:

- a periodically-extended version (periodic in N) of the result of convolving the two signals:
  $(x \circledast h)[n] = \sum_{m=-\infty}^{\infty} (x * h)[n - mN]$

- a convolution of one of the signals with a periodically-extended version of the other:
  $(x \circledast h)[n] = (x * h_p)[n]$, where $h_p[n] = \sum_{m=-\infty}^{\infty} h[n - mN]$