6.003: Signal Processing

good morning! we will start @ 8:05 am Eastern

Convolution and Filtering

time domain

$$y(t) = (h * x)(t) = \int h(\tau)x(t-\tau)\,d\tau$$

$$y[n] = (h * x)[n] = \sum_m h[m]x[n-m]$$

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$$

$$\underline{(x*h)[n]}$$

frequency domain

$$Y(\omega) = H(\omega)X(\omega)$$

$$Y(\Omega) = H(\Omega)X(\Omega)$$

$$X(\Omega) \rightarrow \boxed{H(\Omega)} \rightarrow Y(\Omega)$$

$$\underline{X(\Omega)H(\Omega)}$$

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Filtering

We can view filtering in both the time and frequency domains:

Time Domain:

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = (h * x)[n]$$

Frequency Domain:

scale, phase shift

$$X(\Omega) \longrightarrow \boxed{H(\Omega)} \longrightarrow \underline{Y(\Omega)} = \overbrace{\underline{H(\Omega)}}\,\overbrace{\underline{X(\Omega)}}$$

Each frequency component $X(\Omega)$ is scaled by a factor $H(\Omega)$, which can be possibly complex.

The system is completely described by the set of scale factors $H(\cdot)$, which we refer to as the **frequency response** of the system.

$$y[n] = (x * h)[n]$$

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} y[n] e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[m] h[n-m] e^{-j\Omega n}$$

swap order

pull this out of inner sum

$$= \sum_{m=-\infty}^{\infty} x[m] \sum_{n=-\infty}^{\infty} h[n-m] e^{-j\Omega n}$$

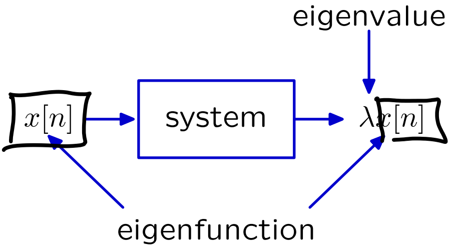
now, let $p = n - m$

$$= \sum_{m=-\infty}^{\infty} x[m] \sum_{p=-\infty}^{\infty} h[p] e^{-j\Omega p} \underbrace{e^{-j\Omega m}}_{\text{pull this out}}$$

$$= \boxed{\sum_{m=-\infty}^{\infty} x[m] e^{-j\Omega m}} \boxed{\sum_{p=-\infty}^{\infty} h[p] e^{-j\Omega p}} = X(\Omega) \cdot H(\Omega)$$

Eigenfunctions and Eigenvalues

If the output signal is a scalar multiple of the input signal, we refer to the signal as an **eigenfunction**, and the multiplier as the **eigenvalue**.



Complex Exponentials

Complex exponentials are eigenfunctions of LTI systems. If $h[\cdot]$ is a system's unit sample response, and $x[n] = e^{j\Omega n}$, then the system's output is:

$$e^{j\Omega n} \rightarrow \boxed{h[n]} \rightarrow$$

$$y[n] = \sum_{m=-\infty}^{\infty} h[m] \underbrace{e^{j\Omega(n-m)}}_{e^{j\Omega n} e^{-j\Omega m}}$$

$$= e^{j\Omega n} \underbrace{\sum_{m=-\infty}^{\infty} h[m] e^{-j\Omega m}}$$

$$= \underbrace{e^{j\Omega n}} \underbrace{H(\Omega)} = |H(\Omega)| e^{j(\Omega n + \angle H(\Omega))}$$

Complex Exponentials

Complex exponentials are eigenfunctions of LTI systems.

$$e^{j\Omega n} \longrightarrow \boxed{\text{system}} \longrightarrow H(\Omega)e^{j\Omega n}$$

The eigenvalues $H(\Omega)$ are generally complex-valued, and so they affect both the amplitude and phase of the output:

$$e^{j\Omega n} \longrightarrow \boxed{\text{system}} \longrightarrow |H(\Omega)|e^{j(\Omega n+\angle H(\Omega))}$$

Response to Eternal Sinusoids

If $h[n]$ is purely real, we have $H(-\Omega) = H^*(\Omega)$.

Consider $x[n] = \cos(\Omega n)$ (for all n), which can be written as:

$$x[n] = \frac{1}{2}\left(e^{j\Omega n} + e^{-j\Omega n}\right)$$

$$\cos(\Omega n) \rightarrow \boxed{h[n]} \rightarrow$$

Then:

$$y[n] = \frac{1}{2}\left(\underline{e^{j\Omega n} H(\Omega)} + \underline{e^{-j\Omega n} H^*(\Omega)}\right)$$

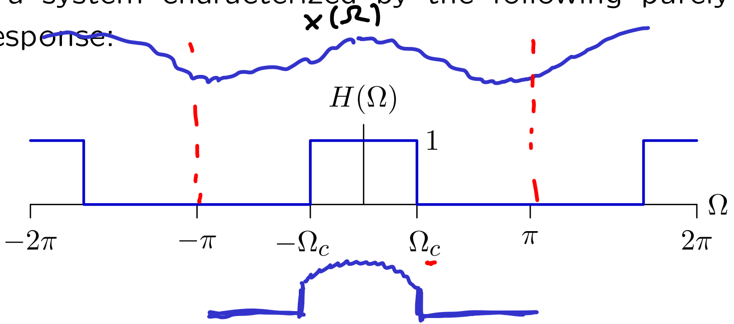
$$= \mathrm{Re}\left(e^{j\Omega n} \underline{H(\Omega)}\right)$$

$$= \mathrm{Re}\left(|H(\Omega)| e^{j(\Omega n + \angle H(\Omega))}\right)$$

$$= |H(\Omega)| \cos(\Omega n + \angle H(\Omega))$$

The "Ideal" Low-Pass Filter

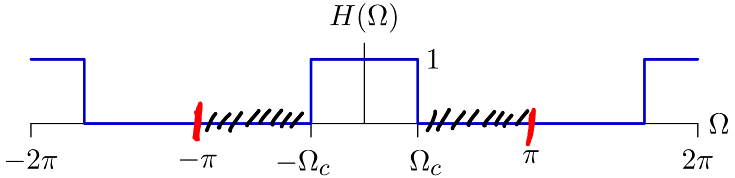
Consider a system characterized by the following purely real frequency response:



Such a system is called a **low-pass filter**, because it allows low frequencies to pass through unmodified, while attenuating high frequencies.

We could apply this filter to a signal by multiplying the DTFT of that signal by the values above. But we could also apply the filter by operating in the time domain. $(x * h)[n]$

The "Ideal" Low-Pass Filter



We can apply this filter to a signal by convolving with its unit sample response. What is the unit sample response of the system whose frequency response is shown above?

unit sample resp   freq resp

$$h \xleftrightarrow{\text{DTFT}} H$$

$$(x * h)[n]$$

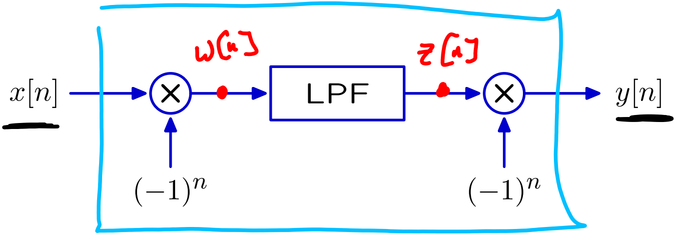
$$h[n] = \frac{1}{2\pi}\int_{2\pi} H(\Omega)e^{j\Omega n}\,d\Omega = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{j\Omega n}\,d\Omega$$

$$= \frac{1}{2\pi}\,\frac{1}{jn}\,e^{j\Omega n}\Big|_{\Omega=-\Omega_c}^{\Omega_c} = \frac{1}{\cancel{2}\pi \cancel{j} n}\overbrace{\left(e^{j\Omega_c n} - e^{-j\Omega_c n}\right)}^{\cancel{2j}\sin(\Omega_c n)}$$

$$\frac{\sin(\Omega_c n)}{\pi n}$$

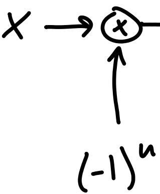
Cascaded System

Consider this system, where LPF represents a lowpass filter of the form discussed on the previous slides.



How many of the following statements are true?

- The transformation from $x[n]$ to $y[n]$ is linear.
- The transformation from $x[n]$ to $y[n]$ is time invariant.
- The transformation from $x[n]$ to $y[n]$ is a high-pass filter.



linear? ✓

$$x[n] = a x_1[n] + b x_2[n]$$

$$y[n] \stackrel{?}{=} a y_1[n] + b y_2[n]$$

$$y[n] = (-1)^n \left(a x_1[n] + b x_2[n] \right)$$

$$= a \underbrace{(-1)^n x_1[n]}_{y_1[n]} + b \underbrace{(-1)^n x_2[n]}_{y_2[n]}$$

time-invariant? ✗

$$x[n] = x_1[n-n_0]$$

$$y[n] \stackrel{?}{=} y_1[n-n_0] = (-1)^{n-n_0} x_1[n-n_0]$$

$$= (-1)^n x_1[n-n_0]$$

$$X \rightarrow \boxed{\text{LPF}} \rightarrow Y$$

$$Y(\Omega) = X(\Omega)\, H(\Omega)$$

$$\Downarrow$$

$$y[n] = (x * h)[n]$$

Convolution is linear
time-invariant

(proof on next 2 pages)

linearity of convolution

$$x[n] = a x_1[n] + b\, x_2[n]$$

$$(x * h)[n] \stackrel{?}{=} a(x_1 * h)[n] + b(x_2 * h)[n]$$

$$(x * h)[n] = \sum_{m=-\infty}^{\infty} \left(a x_1[m] + b x_2[m] \right) h[n-m]$$

distribute

$$= \sum_{m=-\infty}^{\infty} \left(a x_1[m] h[n-m] + b x_2[m] h[n-m] \right)$$

break apart sum

$$= a \sum_{m=-\infty}^{\infty} x_1[m] h[n-m] + b \sum_{m=-\infty}^{\infty} x_2[m] h[n-m]$$

$$= a(x_1 * h)[n] + b(x_2 * h)[n] \quad \checkmark$$

convolution is linear

time-invariance of convolution

$$x[n] = x_1[n-n_0]$$

$$(x*h)[n] \stackrel{?}{=} (x_1*h)[n-n_0]$$

$$(x*h)[n] = \sum_{m=-\infty}^{\infty} x_1[m-n_0]\, h[n-m]$$

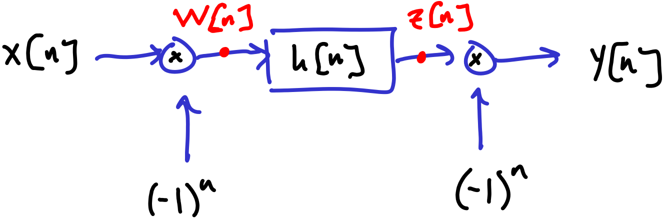
let $p = m - n_0$

$n-m = n-(p+n_0) = (n-n_0)-p$

$$= \sum_{p=-\infty}^{\infty} x_1[p]\, h[(n-n_0)-p]$$

$$= (x_1*h)[n-n_0]$$ ✓ convolution is time-invariant

now back to the big system on the next few slides



$$(-1)^{n+m} = (-1)^{n-m}$$

$$w[n] = (-1)^n x[n]$$

$$z[n] = (w * h)[n] = \sum_{m=-\infty}^{\infty} (-1)^m x[m]\, h[n-m]$$

$$y[n] = (-1)^n z[n]$$

$$= (-1)^n \sum_{m=-\infty}^{\infty} (-1)^m x[m]\, h[n-m]$$

$$= \sum_{m=-\infty}^{\infty} (-1)^{n+m} x[m]\, h[n-m]$$

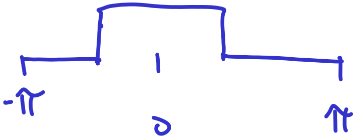
$$g[n] = (-1)^n h[n]$$

$$= \sum_{m=-\infty}^{\infty} x[m] \left((-1)^{n-m} h[n-m] \right) = (x * g)[n]$$

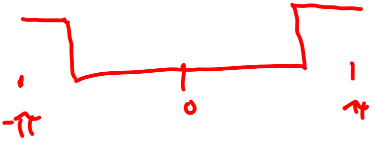
$$y[n] = (x * g)[n]$$

$$Y(\Omega) = X(\Omega) G(\Omega)$$

$H(\Omega)$



$G(\Omega)$



high-pass filter

$$g[n] = (-1)^n h[n]$$

$$\Downarrow \quad \underline{e^{j\pi n}}$$

$$G(\Omega) = \sum_{n=-\infty}^{\infty} (-1)^n h[n] e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} e^{j\pi n} h[n] e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} h[n] e^{-j(\Omega - \pi) n}$$

$$= \underline{H(\Omega - \pi)}$$