So far, we have focused our attention on signals and their mathematical representations. However, we are often also interested in manipulating signals. To this end, we introduce the notion of a system (or filter).

Examples:
- audio enhancement: equalization, noise reduction, reverberation, echo cancellation, pitch shift (auto-tune)
- image enhancement: smoothing, edge enhancement, unsharp masking, feature detection
- video enhancement: image stabilization, motion magnification
Representations of Signals

Characterize systems by their input/output relationships.

- **Difference Equation**: Represent system by algebraic constraints on samples.
- **Convolution**: Represent system by its unit sample response.
- **Filter**: Represent system as amplification/attenuation of frequency components.
A system is linear if its response to a **weighted sum of inputs** is equal to the **weighted sum of its responses** to each of the inputs.

Given

\[ x_1[n] \rightarrow \text{system} \rightarrow y_1[n] \]

and

\[ x_2[n] \rightarrow \text{system} \rightarrow y_2[n] \]

the **system is linear** if

\[ \alpha x_1[n] + \beta x_2[n] \rightarrow \text{system} \rightarrow \alpha y_1[n] + \beta y_2[n] \]

is true for all \( \alpha \) and \( \beta \) and all possible inputs.

A system is linear if it is both additive and homogeneous.
A system is time-invariant if delaying the input to the system simply delays the output by the same amount of time.

Given

\[ x[n] \rightarrow \text{system} \rightarrow y[n] \]

the **system is time invariant** if

\[ x[n-n_0] \rightarrow \text{system} \rightarrow y[n-n_0] \]

is true for all \( n_0 \) and for all possible inputs.
If a system is linear and time-invariant, its input-output relation is completely specified by the system’s *unit sample response* \( h[n] \).

![Diagram showing system input and output](image)

The unit sample response \( h[n] \) is the output of the system when the input is the unit sample signal \( \delta[n] \).

The output for more complicated inputs can be computed by summing scaled and shifted versions of the unit sample response.
Consider the following signal:

\[
    x[n] = \begin{cases} 
        1 & \text{if } n = 0 \\
        -1 & \text{if } n = 3 \\
        -2 & \text{if } n = 4 \\
        0 & \text{otherwise} 
    \end{cases}
\]

This signal can be represented as:

\[
    x[n] = \delta[n] - \delta[n-3] - 2\delta[n-4]
\]

In general, we can represent a signal as a sum of scaled, shifted deltas:

\[
    x[n] = \sum_{m=-\infty}^{\infty} x[m] \delta[n-m]
\]

\[
= \ldots + x[-1] \delta[n+1] + x[0] \delta[n] + x[1] \delta[n-1] + x[2] \delta[n-2] + \ldots
\]
In general, we can represent a signal as a sum of scaled, shifted deltas:

\[ x[n] = \sum_{m=-\infty}^{\infty} x[m] \delta[n - m] \]

\[ = \ldots + x[-1] \delta[n + 1] + x[0] \delta[n] + x[1] \delta[n - 1] + x[2] \delta[n - 2] + \ldots \]

If \( h[\cdot] \) is the unit sample response of an LTI system, then the output of that system in response to this arbitrary input \( x[\cdot] \) can be viewed as a sum of scaled, shifted unit sample responses:

\[ y[n] = \sum_{m=-\infty}^{\infty} x[m] h[n - m] \]

If a system is linear and time-invariant (LTI) then its output is the sum of weighted and shifted unit sample responses.

\[ x[n] = \sum_{m=-\infty}^{\infty} x[m] \delta[n - m] \]

\[ y[n] = \sum_{m=-\infty}^{\infty} x[m] h[n - m] \]
Superposition by Example

Consider an LTI system described by \( y[n] = 3x[n] - x[n-1] \). Compute the response of this system to an input \( x[·] \) given by:

\[
x[n] = 2\delta[n-1] + 5\delta[n-2] + 3\delta[n-4]
\]

\[
y[n] = 2h[n-1] + 5h[n-2] + 3h[n-4]
\]

\[
h[n] = 3\delta[n] - \delta[n-1]
\]

<table>
<thead>
<tr>
<th>( n )</th>
<th>(-2)</th>
<th>(-1)</th>
<th>(0)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x[n] )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>5</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( 2h[n-1] )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( 5h[n-2] )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>15</td>
<td>-5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( 3h[n-4] )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>9</td>
<td>-3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( y[n] )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>13</td>
<td>-5</td>
<td>9</td>
<td>-3</td>
<td>0</td>
</tr>
</tbody>
</table>
**Convolution**

Response of an LTI system to an arbitrary input.

\[ y[n] = \sum_{m=-\infty}^{\infty} x[m] h[n - m] \equiv (x * h)[n] \]

This operation is called **convolution** (verb form: **convolve**).
Convolution

Convolution is represented with an asterisk.

\[ \sum_{m=-\infty}^{\infty} x[m] h[n - m] \equiv (x * h)[n] \]

Convolution operates on \textit{signals}, not samples.

The symbols \( x \) and \( h \) represent DT signals.
Convolving \( x \) with \( h \) generates a new DT signal \( x * h \).
Unit sample response $h[n]$ is a complete description of an LTI system.

\[
x[n] \rightarrow h[n] \rightarrow y[n]
\]

Given $h[\cdot]$, we can compute the response $y[\cdot]$ to any input $x[\cdot]$ by convolving $x[\cdot]$ and $h[\cdot]$:

\[
y[n] = (x * h)[n] \equiv \sum_{m=-\infty}^{\infty} x[m]h[n - m]
\]
Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

Start with 3 barrels of wine: newest at left, oldest at right.
Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

Sell half of the oldest stock.

\[-1 \rightarrow -2 \rightarrow -3 \rightarrow -3\] sell
Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

Refill oldest barrel from next-to-oldest barrel.
Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

Refill next-to-oldest barrel from youngest barrel.

![Diagram of Solera Process]
Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

Refill youngest barrel with this year’s harvest.

![Diagram showing the Solera process with barrels labeled 0, -1, -2, and -3, with an arrow indicating the flow from 0 to -1, and a note to sell at -3.]
Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

Old and new contents mix; ready for next year.

\[
\begin{array}{ccc}
0 & \quad & -1 \\
-1 & \quad & -2 \\
-2 & \quad & -3
\end{array}
\]
Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

Old and new contents mix; ready for next year.

Properties of solera process:
- Mixing produces a more uniform product.
- Mitigates worst-case results of one bad year.
- Blends wines from MANY previous years.
Solera Analysis

We can analyze these effects with a tracer experiment.

Add 1 unit of tracer to new crop; track tracer through the system.

How much tracer will be in each barrel at the end of year 3?
Add 1 unit of tracer to new crop; track tracer through the system.

<table>
<thead>
<tr>
<th>Year</th>
<th>Tracer in</th>
<th>Barrel #1</th>
<th>Barrel #2</th>
<th>Barrel #3</th>
<th>Tracer out</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1/4</td>
<td>2/4</td>
<td>1/4</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1/8</td>
<td>3/8</td>
<td>3/8</td>
<td>1/8</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>1/16</td>
<td>4/16</td>
<td>6/16</td>
<td>3/16</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>1/32</td>
<td>5/32</td>
<td>10/32</td>
<td>6/32</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>1/64</td>
<td>6/64</td>
<td>15/64</td>
<td>10/64</td>
</tr>
</tbody>
</table>

$x[n]$ and $y[n]$ graphs are shown in the image. The $x[n]$ graph shows the tracer added to the new crop, and the $y[n]$ graph shows the tracer exiting the system over time.
How would results change if tracer were added in year 1 (not 0)?

Original response:

\[
\begin{align*}
\text{Delayed input} & \rightarrow \text{delayed output:} \\
\rightarrow x[n] & \\
1 & \\
\rightarrow y[n] & \\
3/16 & \\
\end{align*}
\]

Delaying the input by a year simply delays the outputs by one year.
Time-Invariance

A system is time-invariant if delaying the input to the system simply delays the output by the same amount of time.

Given

\[ x[n] \rightarrow \text{system} \rightarrow y[n] \]

the system is **time invariant** if

\[ x[n - n_0] \rightarrow \text{system} \rightarrow y[n - n_0] \]

is true for all \( n_0 \).
Scaling the input amplitudes:

\[ 0.5x[n] \]

\[ 0.5y[n] \]

Adding two inputs:

\[ x[n] + x[n-6] \]

\[ y[n] + y[n-6] \]

Linearly combining two inputs:

\[ x[n] + 0.5x[n-6] \]

\[ y[n] + 0.5y[n-6] \]
Linearity

A system is linear if its response to a weighted sum of inputs is equal to the weighted sum of its responses to each of the inputs.

Given

- $x_1[n] \xrightarrow{\text{system}} y_1[n]$ and
- $x_2[n] \xrightarrow{\text{system}} y_2[n]$,

the system is linear if

- $\alpha x_1[n] + \beta x_2[n] \xrightarrow{\text{system}} \alpha y_1[n] + \beta y_2[n]$ is true for all $\alpha$ and $\beta$. 
Convolution

If a system is linear and time invariant, the response to an arbitrary input is the convolution of that input with the unit-sample response.

\[
\begin{align*}
\delta[n] & \xrightarrow{\text{system}} h[n] \\
\delta[n-m] & \xrightarrow{\text{system}} h[n-m] \\
x[k]\delta[n-m] & \xrightarrow{\text{system}} x[k]h[n-m] \\
x[n] &= \sum_{m=-\infty}^{\infty} x[k]\delta[n-m] \\
y[n] &= \sum_{m=-\infty}^{\infty} x[k]h[n-m]
\end{align*}
\]
Check Yourself

For solera process ...

<table>
<thead>
<tr>
<th>Year</th>
<th>Tracer in</th>
<th>Barrel #1</th>
<th>Barrel #2</th>
<th>Barrel #3</th>
<th>Tracer out</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>x[n]</td>
<td></td>
<td></td>
<td></td>
<td>y[n]</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1/4</td>
<td>2/4</td>
<td>1/4</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1/8</td>
<td>3/8</td>
<td>3/8</td>
<td>1/8</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>1/16</td>
<td>4/16</td>
<td>6/16</td>
<td>3/16</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>1/32</td>
<td>5/32</td>
<td>10/32</td>
<td>6/32</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>1/64</td>
<td>6/64</td>
<td>15/64</td>
<td>10/64</td>
</tr>
</tbody>
</table>

How much tracer would be found in output from year 5, if one unit of tracer is added in years 0, 1, and 2?

1. \( \frac{21}{32} \)
2. \( \frac{1}{2} \)
3. \( \frac{3}{16} \)
4. \( \frac{9}{16} \)
5. none of above
Divide and Conquer

The content of barrel #3 has no direct dependence on barrel #1.
The new content of barrel #3 depends only on itself and barrel #2.
All dependence on barrel #1 is through barrel #2.

Since barrel #3 depends only on barrel #2, and barrel #2 depends
only on barrel #1, the three barrel system is equivalent to the cas-
cade of three one barrel systems!
## Divide and Conquer

Making a three-barrel system by cascading three one-barrel systems.

![Diagram](attachment:image.png)

<table>
<thead>
<tr>
<th>Year</th>
<th>Tracer in $x[n] = \delta[n]$</th>
<th>Barrel 1</th>
<th>Barrel 2</th>
<th>Barrel 3</th>
<th>Tracer out $y[n] = h_3[n]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1/4</td>
<td>2/4</td>
<td>1/4</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1/8</td>
<td>3/8</td>
<td>3/8</td>
<td>1/8</td>
</tr>
<tr>
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<td>0</td>
<td>1/16</td>
<td>4/16</td>
<td>6/16</td>
<td>3/16</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>1/32</td>
<td>5/32</td>
<td>10/32</td>
<td>6/32</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>1/64</td>
<td>6/64</td>
<td>15/64</td>
<td>10/64</td>
</tr>
</tbody>
</table>

$x[n] = \delta[n]$

$y[n] = h_3[n]$
Divide and Conquer

Find the unit-sample response $h_1[n]$ for a 1-barrel solera.
Find the unit-sample response $h_1[n]$ for a 1-barrel solera.

$$h_1[n] = \left(\frac{1}{2}\right)^n \delta[n-1]$$
Divide and Conquer

How can we use this to find \( h_2[n] \) (output of a two-barrel system)?

\[
S[n] \rightarrow \text{barrel} \rightarrow h_1[n] \rightarrow \text{barrel} \rightarrow h_2[n]
\]

Second barrel identical to first (unit sample response = \( h_1[n] \))

\( h_2[n] = (h_1 * h_1)[n] \)

(Work and answer on next page)
\[ h_2[n] = (h_1 * h_1)[n] = \sum_{m=-\infty}^{\infty} h_1[m] h_1[n-m] \]

**Definition of Convolution**

Plug in \( h_1[0] \):

\[ \sum_{m=-\infty}^{\infty} \left( \frac{1}{2} \right)^m u[m-1] \left( \frac{1}{2} \right)^{n-m} u[n-m-1] \]

- O unless \( m \geq 1 \)
- O unless \( m \leq n-1 \)

These constraints our sum! The terms in the sum are only nonzero when \( 1 \leq m \leq n-1 \) (also implies \( n \geq 2 \))

\[ \sum_{m=1}^{n-1} \left( \frac{1}{2} \right)^m (1) \left( \frac{1}{2} \right)^{n-m} (1) = \sum_{m=1}^{n-1} \left( \frac{1}{2} \right)^n \]

\[ = \frac{1}{2^n} \sum_{m=1}^{n-1} (1) = (n-1)\left( \frac{1}{2} \right)^n u[n-2] \]

- Doesn't depend on \( m \)!

To account for \( h_2[n \leq 0] = 0 \)
Divide and Conquer

How can we use this to find $h_3[n]$ (output of a three-barrel system)?

$\delta[n] \rightarrow \begin{array}{c} \text{barrel} \\ h_1 \end{array} \rightarrow \begin{array}{c} \text{barrel} \\ h_2 \end{array} \rightarrow \begin{array}{c} \text{barrel} \\ h_3 \end{array} \rightarrow h_3[n]$

notice:

$\begin{array}{c} \text{barrel} \\ h_2[n] \end{array} \rightarrow h_3[n]$

our 1-barrel system from before!

So $h_3[n] = (h_2 \ast h_1)[n]$

(work and answer on following pages)
\[ h_1[n] = \left(\frac{1}{2}\right)^n u[n-1] \quad h_2[n] = (n-1)\left(\frac{1}{2}\right)^n u[n-2] \]

\[ h_3[n] = (h_1 * h_2)[n] = \sum_{m=-\infty}^{\infty} h_1[m] h_2[n-m] \]

Plug in \( h_1 \) and \( h_2 \)

\[ = \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m u[m-1] (n-m-1)\left(\frac{1}{2}\right)^{n-m} u[n-m-2] \]

Terms are all 0, except when \( m \geq 1 \) and \( n-m \geq 2 \)

(Where \( 1 \leq m \leq n-2 \), which also implies \( n \geq 3 \))

\[ = \sum_{m=1}^{n-2} (n-m-1)\left(\frac{1}{2}\right)^n \]

\[ = \left(\frac{1}{2}\right)^n \sum_{m=1}^{n-2} (n-m-1) \]

This looks ok, but how to simplify the sum? If we don't remember a formula for this, that's ok, we can think! (see next page)
\[ h_3[n] = \left( \frac{1}{2} \right)^{n-2} \sum_{m=1}^{n-2} (n-m-1) \]

Let's see how we can simplify

\[ \sum_{m=1}^{n-2} (n-m-1) = n-2 + n-3 + n-4 + n-5 + \ldots + n-(n-2)-1 \]

we have \( n-2 \) terms like \( n \), so \( = n-(n-1) = 1 \)

Separate those out

\[ = (n-2)n - \sum_{i=2}^{n-1} i \]

\[ = 2 + 3 + 4 + \ldots + n-1 \]

\[ \sum_{i=2}^{n-1} i = \frac{n(n-1)}{2} \]

Regroup to:

\[ = 2 + 3 + 4 + 5 + \ldots + (n-1) + n \]

And we have \( \frac{n-2}{2} \) of these terms!

\[ = \left( \frac{n-2}{2} \right) (n+1) \]

Now plug in:

\[ = 2n^2 - 4n - \left( \frac{n^2 - n - 2}{2} \right) \]

\[ = \frac{n^2 - 3n + 2}{2} = \frac{(n-1)(n-2)}{2} \]

Hooray!

\[ \frac{(n-2)}{2} (n+1) \]

Finish up on next slide.
\[ h_3^n = \left( \frac{1}{2} \right)^{n-2} \sum_{m=1}^{n-2} (n-m-1) = \left( \frac{1}{2} \right)^{n-2} \left( \frac{(n-1)(n-2)}{2} \right) u[n-3] \]

(matches graph on page 23, which we got via simulating the whole system)