6.003: Signal Processing

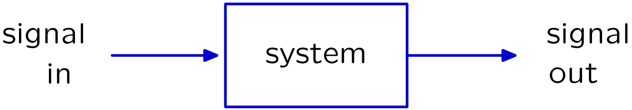
Superposition and Convolution

$$y[n] = (h * x)[n] = \sum_m h[m]x[n-m]$$

30 March 2020

Systems

So far, we have focused our attention on signals and their mathematical representations. However, we are often also interested in *manipulating* signals. To this end, we introduce the notion of a **system** (or **filter**).

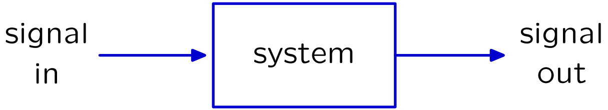


Examples:

- audio enhancement: equalization, noise reduction, reverberation, echo cancellation, pitch shift (auto-tune)
- image enhancement: smoothing, edge enhancement, unsharp masking, feature detection
- video enhancement: image stabilization, motion magnification

Representations of Signals

Characterize systems by their input/output relationships.



Difference Equation: represent system by algebraic constraints on samples

Convolution: represent system by its unit sample response

Filter: represent system as amplfication/attentuation of frequency components

Linearity

A system is linear if its response to a **weighted sum of inputs** is equal to the **weighted sum of its responses** to each of the inputs.

Given

$$x_1[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n]$$

and

$$x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_2[n]$$

the **system is linear** if

$$\alpha x_1[n] + \beta x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow \alpha y_1[n] + \beta y_2[n]$$

is true for all α and β and all possible inputs.

A system is linear if it is both additive and homogeneous.

Time-Invariance

A system is time-invariant if delaying the input to the system simply delays the output by the same amount of time.

Given

$$x[n] \longrightarrow \boxed{\text{system}} \longrightarrow y[n]$$

the **system is time invariant** if

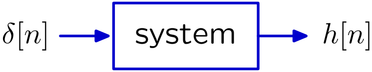
$$x[n-n_0] \longrightarrow \boxed{\text{system}} \longrightarrow y[n-n_0]$$

is true for all n_0 and for all possible inputs.

Unit Sample Response

If a system is linear and time-invariant, its input-output relation is completely specified by the system's **unit sample response** $h[n]$.

The unit sample response $h[n]$ is the output of the system when the input is the unit sample signal $\delta[n]$.



The output for more complicated inputs can be computed by summing scaled and shifted versions of the unit sample response.

Superposition

Consider the following signal:

$$x[n] = \begin{cases} 1 & \text{if } n = 0 \\ -1 & \text{if } n = 3 \\ -2 & \text{if } n = 4 \\ 0 & \text{otherwise} \end{cases}$$

This signal can be represented as:

$$x[n] = \delta[n] - \delta[n-3] - 2\delta[n-4]$$

In general, we can represent a signal as a sum of scaled, shifted deltas:

$$\begin{aligned} x[n] &= \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] \\ &= \ldots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \ldots \end{aligned}$$

Superposition

In general, we can represent a signal as a sum of scaled, shifted deltas:

$$
\begin{aligned}
x[n] &= \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] \\
&= \ldots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \ldots
\end{aligned}
$$

If $h[\cdot]$ is the unit sample response of an LTI system, then the output of that system in response to this arbitrary input $x[\cdot]$ can be viewed as a sum of scaled, shifted *unit sample responses*:

$$
\begin{aligned}
y[n] &= \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\
&= \ldots + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + \ldots
\end{aligned}
$$

Structure of Superposition

If a system is linear and time-invariant (LTI) then its output is the sum of weighted and shifted unit sample responses.

$$\delta[n] \longrightarrow \boxed{\text{system}} \longrightarrow h[n]$$

$$\delta[n-m] \longrightarrow \boxed{\text{system}} \longrightarrow h[n-m]$$

$$x[m]\delta[n-m] \longrightarrow \boxed{\text{system}} \longrightarrow x[m]h[n-m]$$

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] \longrightarrow \boxed{\text{system}} \longrightarrow y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Superposition by Example

Consider an LTI system described by $y[n] = 3x[n] - x[n-1]$. Compute the response of this system to an input $x[\cdot]$ given by:

$$x[n] = 2\delta[n-1] + 5\delta[n-2] + 3\delta[n-4]$$

Convolution

Response of an LTI system to an arbitrary input.

$$x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n]$$

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \equiv (x * h)[n]$$

This operation is called **convolution** (verb form: **convolve**).

Convolution

Convolution is represented with an asterisk.

$$\sum_{m=-\infty}^{\infty} x[m]h[n-m] \equiv (x * h)[n]$$

Convolution operates on **signals**, not samples.

The symbols x and h represent DT signals.

Convolving x with h generates a new DT signal $x * h$.

DT Convolution: Summary

Unit sample response $h[n]$ is a complete description of an LTI system.

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

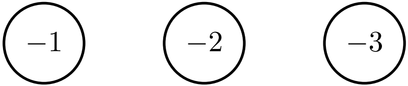
Given $h[\cdot]$, we can compute the response $y[\cdot]$ to any input $x[\cdot]$ by convolving $x[\cdot]$ and $h[\cdot]$:

$$y[n] = (x * h)[n] \equiv \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

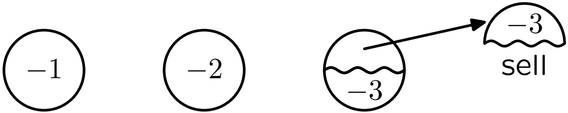
Start with 3 barrels of wine: newest at left, oldest at right.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

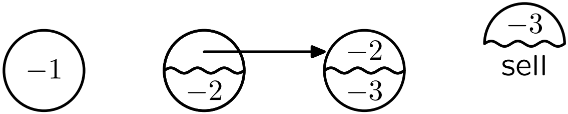
Sell half of the oldest stock.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

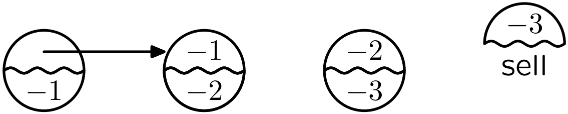
Refill oldest barrel from next-to-oldest barrel.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

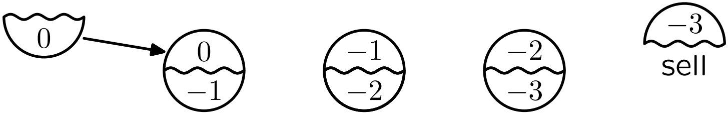
Refill next-to-oldest barrel from youngest barrel.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

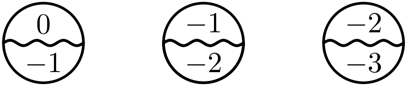
Refill youngest barrel with this year's harvest.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

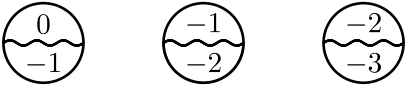
Old and new contents mix; ready for next year.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

Old and new contents mix; ready for next year.



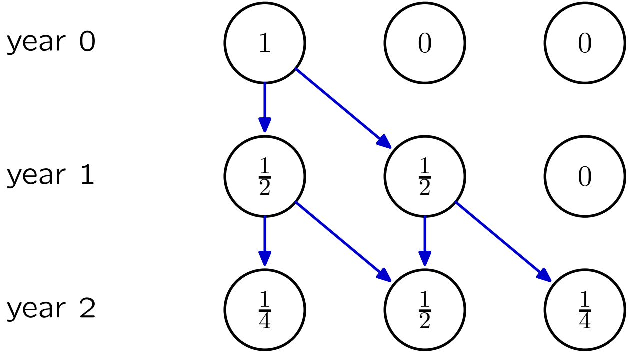
Properties of solera process:

- Mixing produces a more uniform product.
- Mitigates worst-case results of one bad year.
- Blends wines from MANY previous years.

Solera Analysis

We can analyze these effects with a tracer experiment.

Add 1 unit of tracer to new crop; track tracer through the system.

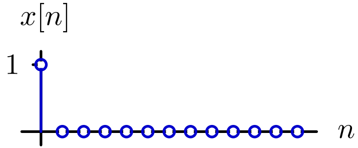


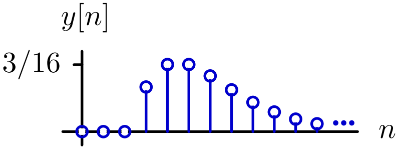
How much tracer will be in each barrel at the end of year 3?

Solera Analysis

Add 1 unit of tracer to new crop; track tracer through the system.

Year n	Tracer in $x[n]$	Barrel #1	Barrel #2	Barrel #3	Tracer out $y[n]$
0	1	1	0	0	0
1	0	1/2	1/2	0	0
2	0	1/4	2/4	1/4	0
3	0	1/8	3/8	3/8	1/8
4	0	1/16	4/16	6/16	3/16
5	0	1/32	5/32	10/32	6/32
6	0	1/64	6/64	15/64	10/64



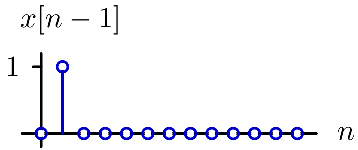


Solera Analysis

How would results change if tracer were added in year 1 (not 0)?

Original response:

Delayed input $\rightarrow$ delayed output:



Delaying the input by a year simply delays the outputs by one year.

Time-Invariance

A system is time-invariant if delaying the input to the system simply delays the output by the same amount of time.

Given

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the system is **time invariant** if

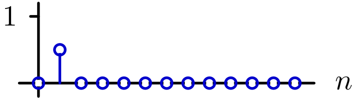
$$x[n-n_0] \longrightarrow \boxed{\text{system}} \longrightarrow y[n-n_0]$$

is true for all n_0.

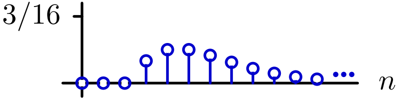
Solera Analysis

Scaling the input amplitudes:

$0.5x[n]$

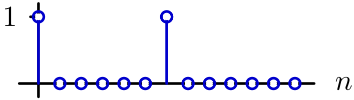


$0.5y[n]$

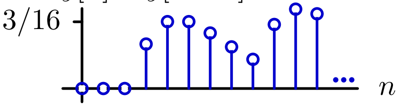


Adding two inputs:

$x[n] + x[n-6]$

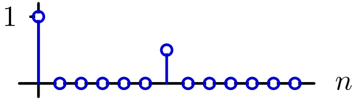


$y[n] + y[n-6]$

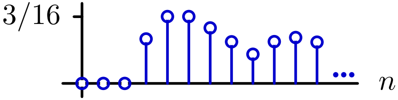


Linearly combining two inputs:

$x[n] + 0.5x[n-6]$



$y[n] + 0.5y[n-6]$



Linearity

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Given

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and

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the system is linear if

$$\alpha x_1[n] + \beta x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow \alpha y_1[n] + \beta y_2[n]$$

is true for all α and β.

Convolution

If a system is linear and time invariant, the response to an arbitrary input is the convolution of that input with the unit-sample response.

$$\delta[n] \longrightarrow \boxed{\text{system}} \longrightarrow h[n]$$

$$\delta[n-m] \longrightarrow \boxed{\text{system}} \longrightarrow h[n-m]$$

$$x[k]\delta[n-m] \longrightarrow \boxed{\text{system}} \longrightarrow x[k]h[n-m]$$

$$x[n] = \sum_{m=-\infty}^{\infty} x[k]\delta[n-m] \longrightarrow \boxed{\text{system}} \longrightarrow y[n] = \sum_{m=-\infty}^{\infty} x[k]h[n-m]$$

Check Yourself

For solera process ...

Year n	Tracer in $x[n]$	Barrel #1	Barrel #2	Barrel #3	Tracer out $y[n]$
0	1	1	0	0	0
1	0	1/2	1/2	0	0
2	0	1/4	2/4	1/4	0
3	0	1/8	3/8	3/8	1/8
4	0	1/16	4/16	6/16	3/16
5	0	1/32	5/32	10/32	6/32
6	0	1/64	6/64	15/64	10/64

How much tracer would be found in output from year 5, if one unit of tracer is added in years 0, 1, and 2?

1. $\frac{21}{32}$ 2. $\frac{1}{2}$ 3. $\frac{3}{16}$ 4. $\frac{9}{16}$ 5. none of above