6.003: Signal Processing

Start @ 8:05am
Eastern

Discrete Fourier Transform (DFT)

Definition and comparison to other Fourier representations.

	analysis	synthesis
DFT:	$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$	$x[n] = \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$
DTFS:	$X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j\frac{2\pi}{N}kn}$	$x[n] = \sum_{k=\langle N \rangle} X[k] e^{j\frac{2\pi}{N}kn}$
DTFT:	$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$	$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$

16 March 2021

Analyzing Frequency Content of Arbitrary Signals

Why use a DFT?

Fourier Series: conceptually simple, but limited to periodic signals.

Fourier Transforms: arbitrary signals, but continuous domain (ω, Ω).

Discrete Fourier Transform: arbitrary DT signals (finite length)

- Discrete in both domains: nice for computation

Today: using the DFT to analyze frequency content of a signal.

Key detection

FFT

The **F**ast **F**ourier **T**ransform is an algorithm for computing the DFT efficiently.

Single Sinusoid

Create ~~three~~ four signals of the following form:

$$x_1[n] = \cos(8\pi n/100)$$

$$x_2[n] = \cos(8\pi n/100 - \pi/4)$$

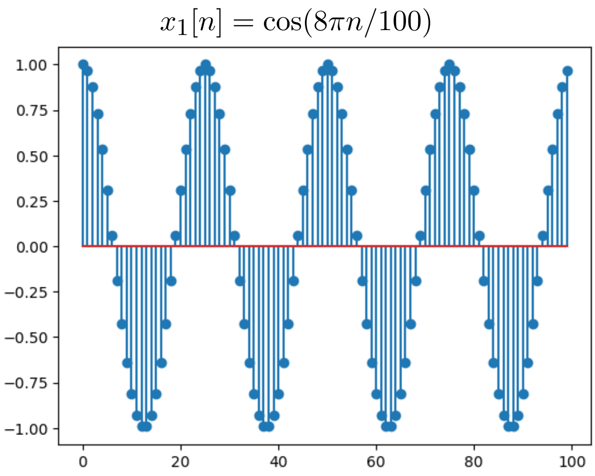
$$x_3[n] = \cos(9\pi n/100)$$

$$x_4[n] = \cos(9\pi n/100 - \pi/2)$$

Each should have a duration of 1 second and should use a sample frequency of 8kHz.

Compare the DFTs of the first 100 samples of each of these signals.

Single Sinusoid



What is the frequency of this tone if the sample rate is 8kHz?

$f = \frac{\Omega f_s}{2\pi}$

$f = \frac{f_s}{N} = \frac{8000}{25} = 320\,\text{Hz}$

Single Sinusoid

Write a program to calculate the DFT of an input sequence.

Use that program to calculate $X_1[k]$, which is the DFT of $x_1[n]$.

$$[x[0],\ x[1],\ x[2],\ \ldots\ x[N-1]]$$

$$\Downarrow$$

$$[X[0],\ X[1],\ X[2],\ \ldots,\ X[N-1]]$$

Single Sinusoid

Write a program to calculate the DFT of an input sequence.

Use that program to calculate $X_1[k]$, which is the DFT of $x_1[n]$.

```
def dft(x):
    N = len(x)
    return [sum(x[n] * e**(-2j*pi*k*n/N) for n in range(N)) / N
            for k in range(N)]

X1 = dft(x[0:100])
```

$$X[k] = \boxed{\frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k n}{N}}}$$

$$x_1[n] = \cos\left(\frac{8\pi}{100}n\right) = \boxed{\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi k}{N}n}}$$

$N=100$

$$\frac{1}{2}e^{j\frac{8\pi}{100}n} + \frac{1}{2}e^{-j\frac{8\pi}{100}n}$$

$$X_1[k] = \begin{cases} 1/2, & k=4 \\ 1/2, & k=-4 \\ 0, & \text{o.w.} \end{cases}$$

Single Sinusoid

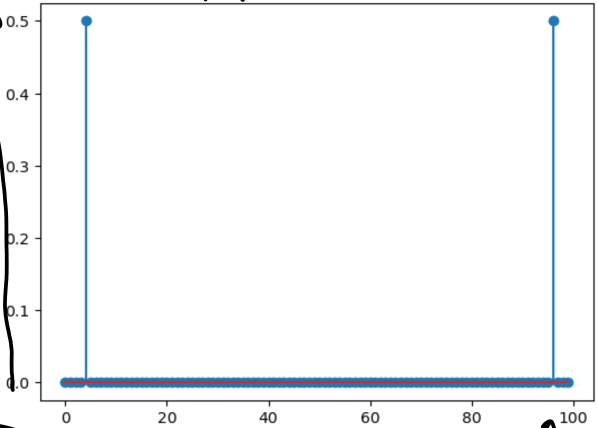
Plot the magnitude of $X_1[\cdot]$.

Single Sinusoid

Plot the magnitude of $X_1[\cdot]$.

$$|X_1[k]| = |DFT\{x_1[0:100]\}|$$

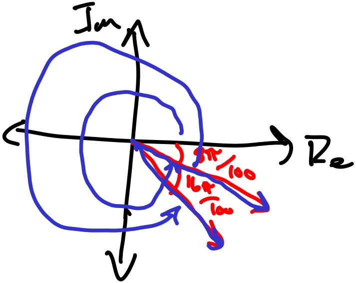
$X_1[k]$



k

4 $\{0, N-1\}$ 96

Which values of k are non-zero?



$$e^{j\frac{2\pi k}{100}n} \rightarrow \boxed{e^{-j\frac{8\pi}{100}n}}$$

$k=-4$

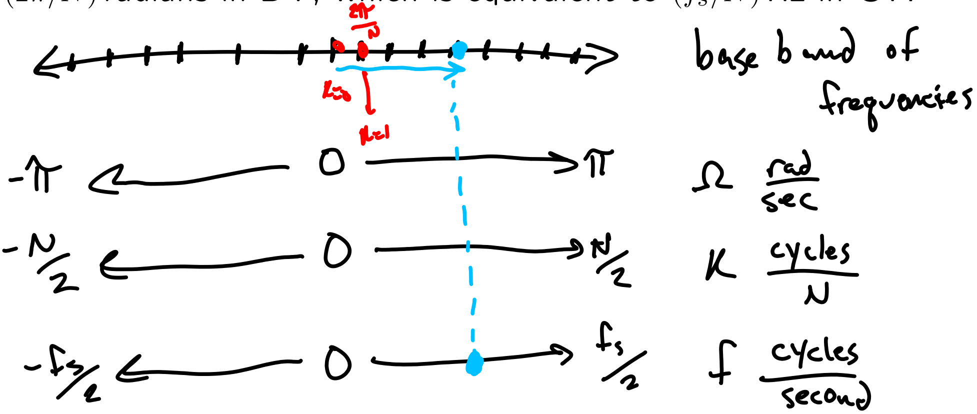
$$\updownarrow$$

$k=96 \rightarrow \boxed{e^{j\frac{192\pi}{100}n}}$

Frequency Scales

k, Ω, f

We can think of the DFT as having spectral resolution of $(2\pi/N)$ radians in DT, which is equivalent to (f_s/N) Hz in CT.

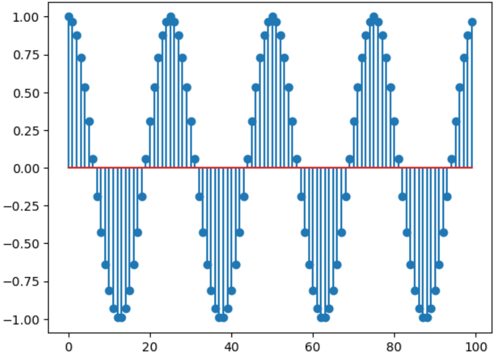


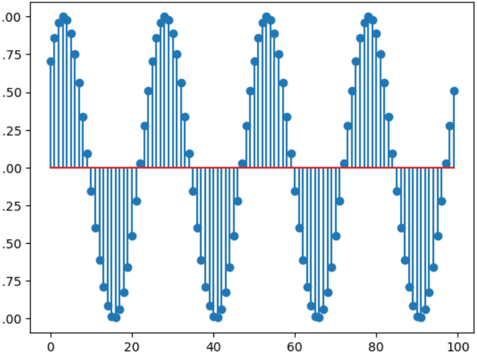
$$\frac{k}{N} = \frac{f}{f_s}$$

Compare Two Signals

How will plots of DFT magnitudes differ for the following signals?

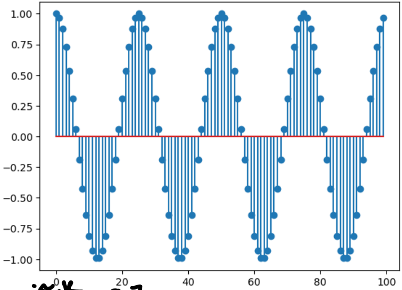
$$x_1[n] = \cos(8\pi n/100) \qquad x_2[n] = \cos(8\pi n/100 - \pi/4)$$



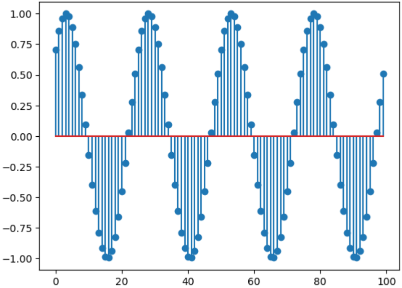


Compare Two Signals

$x_1[n] = \cos(8\pi n/100)$

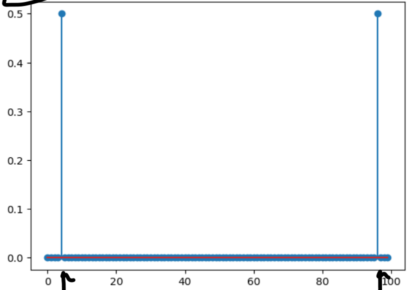


$x_2[n] = \cos(8\pi n/100 - \pi/4)$

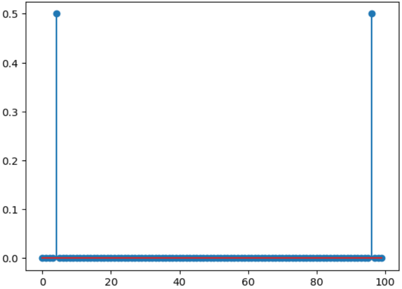


$X_2[k] = \underbrace{e^{-j\pi/4}} X_1[k]$

$|X_1[k]|$

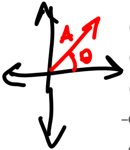


$|X_2[k]|$



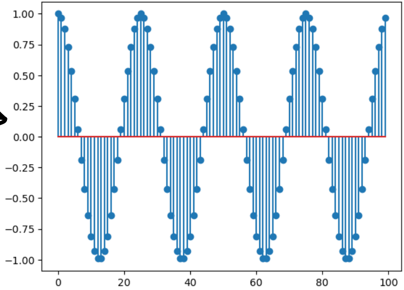
No difference in magnitudes. Are the coefficients actually the same?

Comparing Angles

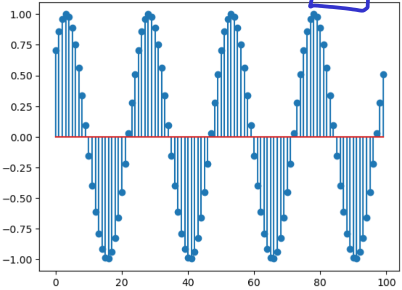


$Ae^{j\theta}$

$x_1[n] = \cos(8\pi n/100)$



$x_2[n] = \cos(8\pi n/100 - \boxed{\pi/4})$

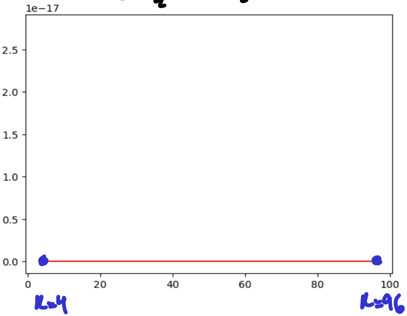


$0e^{j\theta}$

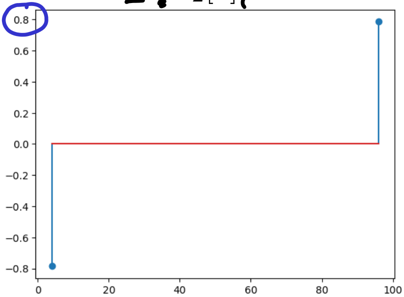
$\Downarrow$

0

$\angle X_1[k]$



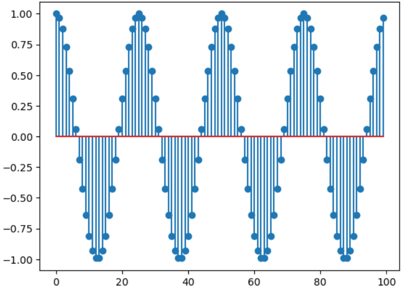
$\angle X_2[k]$



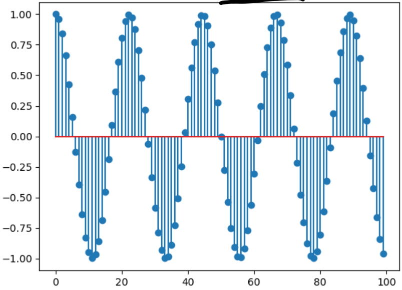
Compare Two Signals

How will plots of DFT magnitudes differ for the following signals?

$$x_1[n] = \cos(\overbrace{8\pi n}/100)$$



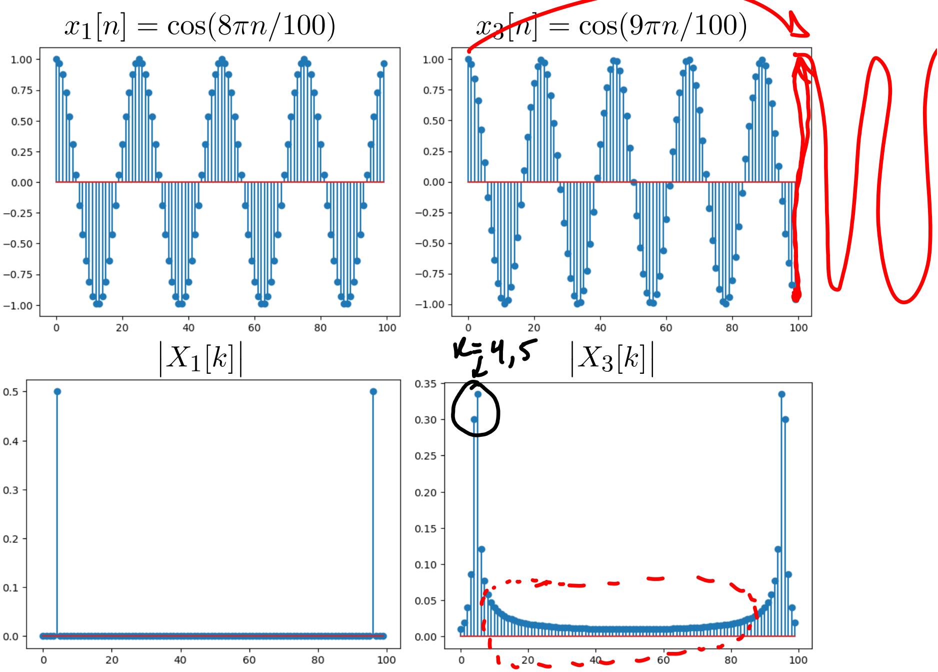
$$x_3[n] = \cos(\overbrace{9\pi n}\underbrace{/100})$$



$k = \pm 4$

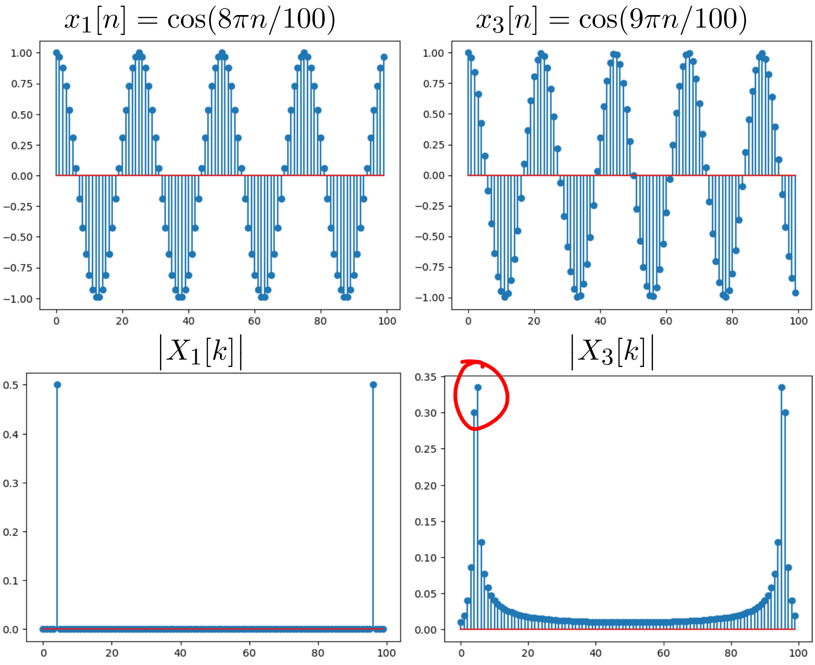
$k \stackrel{?}{=} \pm 4.5$

Compare Two Signals



Why are these DFTs so different?

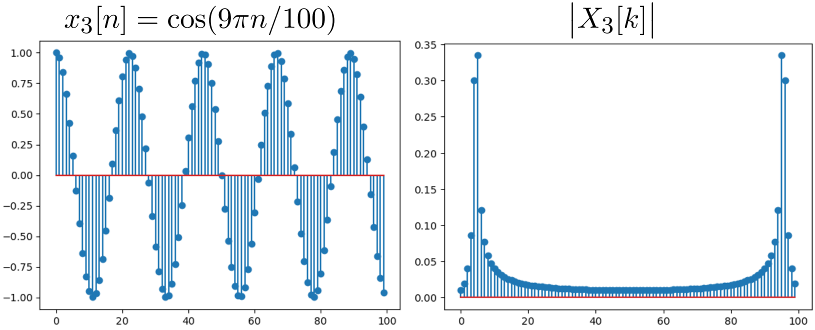
Compare Two Signals



$\Omega_1 \neq \Omega_3$. Even more importantly, $x_3[n]$ is not periodic in $N = 100$!

Single Sinusoid

This blurring occurs because the signal is not periodic in the analysis window ($N = 100$).

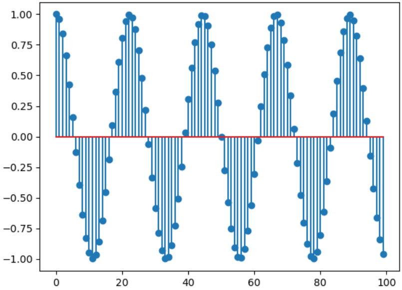


What value of k corresponds to $\Omega = 9\pi/100$?

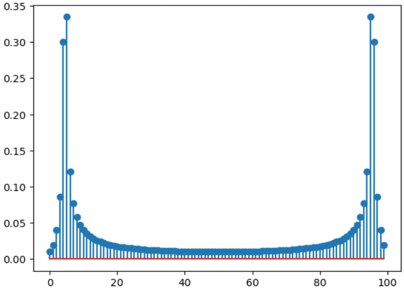
Single Sinusoid

This blurring occurs because the signal is not periodic in the analysis window ($N = 100$).

$x_3[n] = \cos(9\pi n/100)$



$\left|X_3[k]\right|$



What value of k corresponds to $\Omega = 9\pi/100$?

$\Omega = 9\pi/100 = 2\pi k/N$

$k = 4.5$

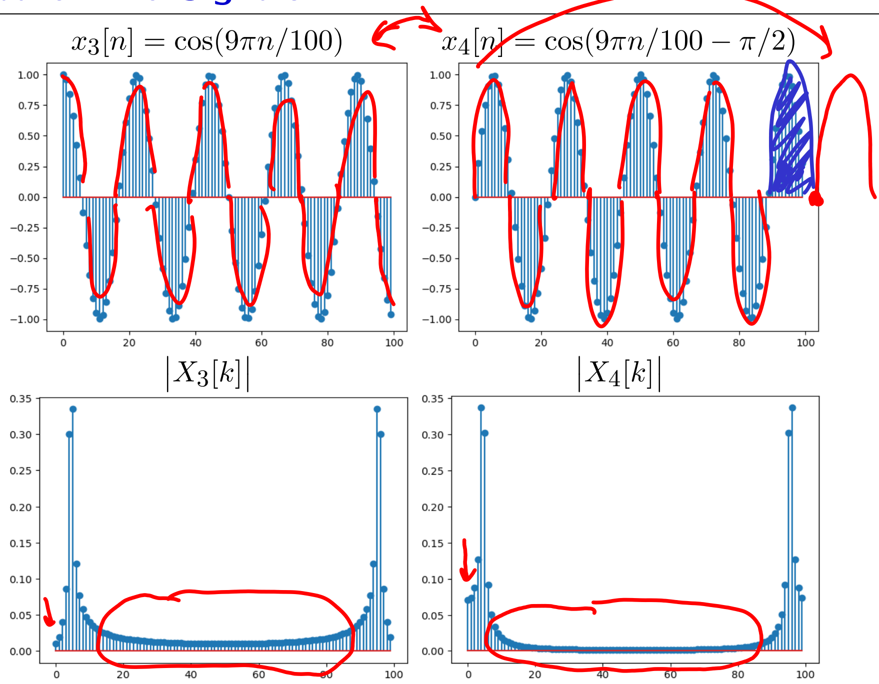
The signal frequency fell between the analysis frequencies.

Compare Two Signals

How will plots of DFT magnitudes differ for the following signals?

- $x_3[n] = \cos(9\pi n/100)$
- $x_4[n] = \cos(9\pi n/100 - \pi/2)$

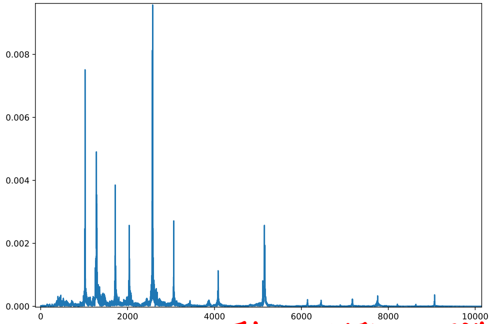
Compare Two Signals



$\Omega_3 = \Omega_4$. But DC bigger. Higher frequencies smaller. Why?

Check Yourself!

For a portion of the Chopin song containing only one chord, 250332 samples long, and recorded with a sampling rate of 48kHz, the DFT magnitudes look like:



e
g
b
e minor

~195Hz ~245Hz ~328Hz

Peaks in magnitude around $k \approx 1021, 1282, 1715, 2037, 2576, 3062, \ldots$

What are the frequencies (in Hz) of the notes being played?

What chord does this correspond to?