6.003: Signal Processing

Fourier Transforms

11 March 2021

Logistics

In response to student feedback and following discussion among the staff, two changes to course policy:

- Starting with PSet 5, feedback about correctness will be shown before the deadline, but you will have a limited number of submissions for each exercise.
- Jing and Adam will be in the Comingle room from 2-3pm on Wednesdays as extra lecture-and-recitation-related office hours (no check-ins)

Today: Fourier Transforms

Last two weeks: representing periodic signals as sums of sinusoids.

This representation provides insights that are not obvious from other representations.

However:

- only works for periodic signals
- must know signal's period *before* doing the analysis

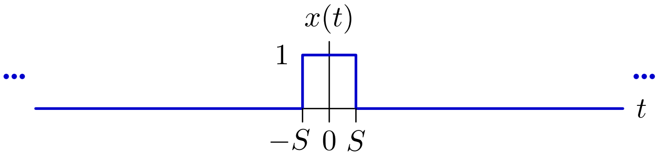
This is impractical (or impossible) for a large category of signals, which we still want to be able to analyze using these methods.

Today

Fourier analysis of *aperiodic* signals: the **Fourier transform**

Fourier Transform

Consider the following (aperiodic) function of time:

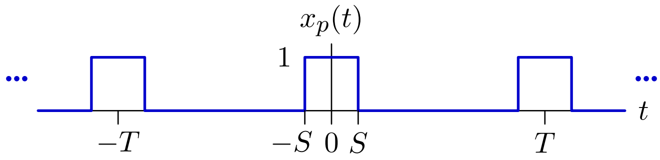


Can we represent it as a sum of sinusoids?

Toward the Fourier Transform

Let's start by considering a related signal $x_p(\cdot)$, which we create by summing shifted copies of $x(\cdot)$:

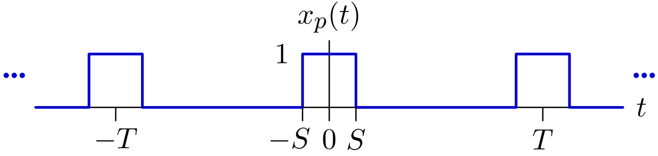
$$x_p(t) = \sum_{m=-\infty}^{\infty} x(t - mT)$$



Now we can directly find the Fourier Series coefficients of this new signal (for arbitrarily-chosen T).

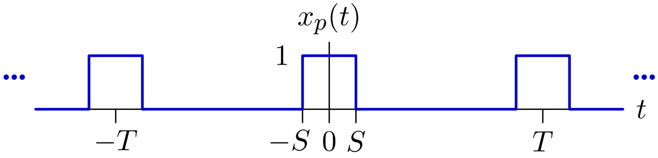
However, maybe that's not really that helpful, since this signal doesn't look much like our original signal $x(\cdot)$. How can we fix that?

Toward the Fourier Transform



This signal doesn't really look much like our original. How can we fix that?

Toward the Fourier Transform



This signal doesn't really look much like our original. How can we fix that?

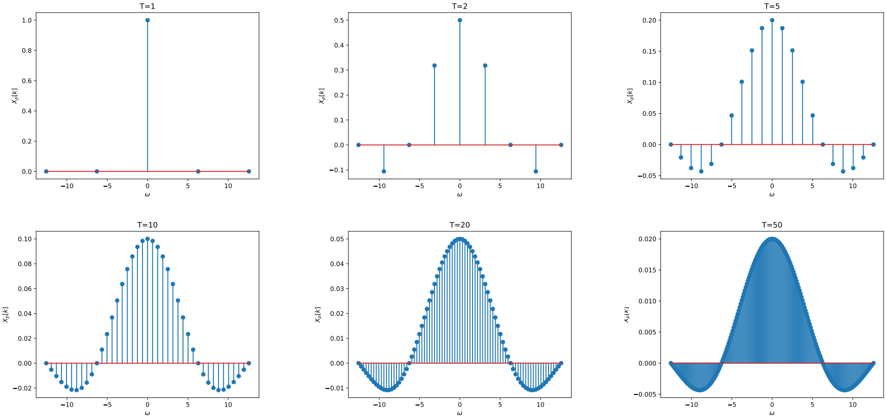
If we let $T \to \infty$, then $x_p(\cdot) \to x(\cdot)$, but $x_p(\cdot)$ is still periodic, so we can still represent it with a Fourier series!

Toward the Fourier Transform

Consequences of $T \to \infty$:

The frequency ω_0 associated with $k = 1$ is defined to be $\frac{2\pi}{T}$. As we increase T, ω_0 gets smaller, and the spacing between the coefficients in terms of rad/sec gets smaller and smaller.

For $S = 0.5$ and for different values of T:



Fourier Series to Fourier Transform

Once we have a periodic signal, we can find the FSC:

$$X_p[k] = \frac{1}{T}\int_T x_p(t)e^{-jk\omega_0 t}dt$$

where $\omega_0 = \frac{2\pi}{T}$.

Now we want to think about $T \to \infty$ Let's replace $\frac{1}{T}$ with $\frac{\omega_0}{2\pi}$, and explicitly pick a period to integrate over:

$$X_p[k] = \frac{\omega_0}{2\pi}\int_{-T/2}^{T/2} x_p(t)e^{-jk\omega_0 t}dt$$

Fourier Series to Fourier Transform

Now, substitute into the synthesis equation:

$$
\begin{aligned}
x_p(t) &= \sum_{k=-\infty}^{\infty} X_p[k] e^{jk\omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} \left\{ \frac{\omega_0}{2\pi} \int_{-T/2}^{T/2} x_p(t) e^{-jk\omega_0 t} dt \right\} e^{jk\omega_0 t}
\end{aligned}
$$

As we take $T \to \infty$, a few things happen:

- $x_p(t) \to x(t)$
- w_0 becomes an infinitesimally small value, $\omega_0 \to \mathrm{d}\omega$
- $k\omega_0$ becomes a continuum, $k\omega_0 \to \omega$ (continuous)
- The bounds of integration approach $-\infty$ and ∞ (respectively)
- The outer sum becomes an integral.

$$
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right\} e^{j\omega t} \mathrm{d}\omega
$$

Fourier Series to Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right\}e^{j\omega t}\mathrm{d}\omega$$

From here, we'll define $X(\omega)$ such that:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

$X(\cdot)$ is the **Fourier Transform** of $x(\cdot)$.

Very similar to the Fourier series, except:

- $x(\cdot)$ need not be periodic
- $x(\cdot)$ can contain *all possible frequencies*

Continuous-Time Fourier Transform

Synthesis Equation

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Analysis Equation

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Continuous-Time Fourier Transform

Synthesis Equation

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Analysis Equation

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Problem: Find the Fourier transform of the following signal.

$$x(t) = e^{-t} u(t) \text{ where } u(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$$

Plot its real and imaginary parts.

Plot its magnitude and phase.

Fourier Transform

Find the Fourier transform of the following signal.

$$x(t) = e^{-t}u(t) \text{ where } u(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$$

Sketch Real and Imaginary Parts

Sketch Magnitude and Phase

Inverse Continuous-Time Fourier Transform

Find the signal whose Fourier transform is

$$X(\omega) = e^{-|\omega|}$$

Continuous-Time Fourier Transform

Find the Fourier transform of:

$x_2(t) = e^{-(t-t_0)}u(t - t_0)$

Continuous-Time Fourier Transform

Find the Fourier transform of:

$x_3(t) = \mathrm{Sym}\{e^{-t}u(t)\}$

Continuous-Time Fourier Transform

Find the Fourier transform of:

$x_4(t) = \mathrm{Asym}\{e^{-t}u(t)\}$

Continuous-Time Fourier Transform

Find the Fourier transform of:

$$x_5(t) = \frac{d}{dt}\mathrm{Sym}\{e^{-t}u(t)\}$$

Continuous-Time Fourier Transform

Operations in time that map to multiplicative factors in frequency:

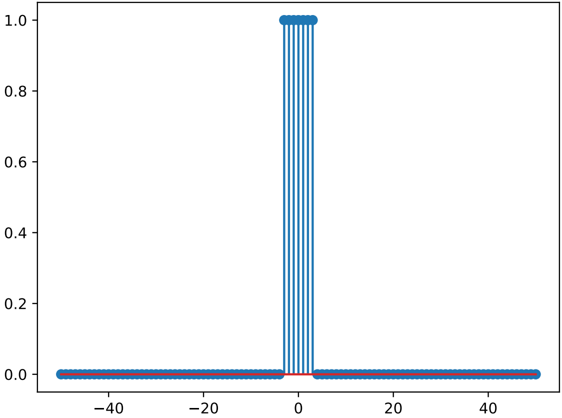
$$x(t) \overset{\text{ctft}}{\longleftrightarrow} X(\omega)$$

$$x(t - t_0) \overset{\text{ctft}}{\longleftrightarrow} e^{j\omega t_0} X(\omega)$$

$$\frac{dx(t)}{dt} \overset{\text{ctft}}{\longleftrightarrow} j\omega X(\omega)$$

Discrete-Time Fourier Transform

We can also apply these same ideas to DT signals. Consider the following (aperiodic) DT signal:

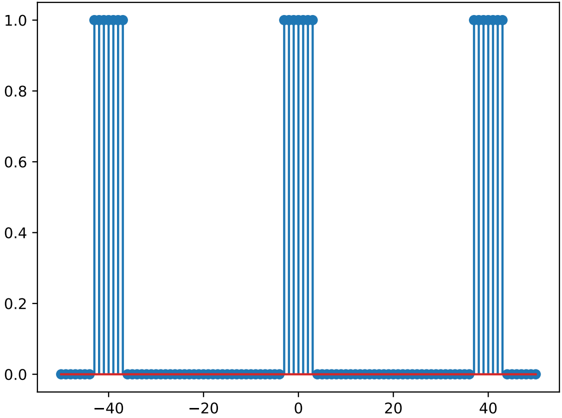


Can we represent this signal as a sum of DT sinusoids?

Toward the DTFT

Start by considering a related signal $x_p[\cdot]$, which we create by summing shifted copies of $x[\cdot]$:

$$x_p[n] = \sum_{m=-\infty}^{\infty} x[n - mN]$$

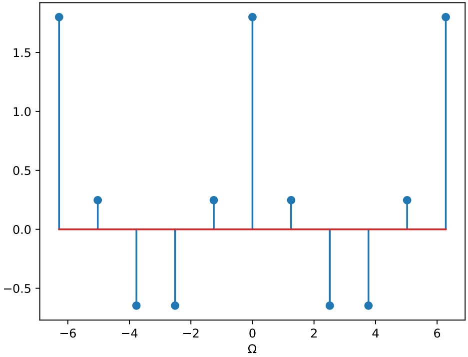


Now we can compute a DTFS (for arbitrarily chosen N)!

Toward the DTFT

If we let $N \to \infty$, then $x_p[\cdot] \to x[\cdot]$, but $x_p[\cdot]$ is still periodic, so we can still represent it with a Fourier series!

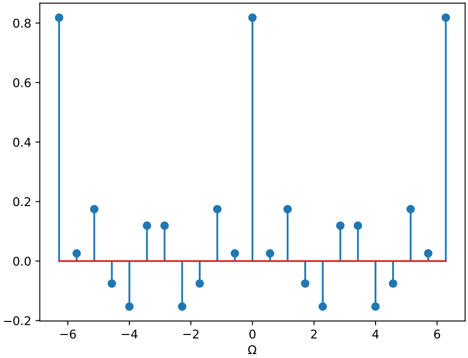
The frequency Ω_0 associated with $k = 1$ is defined to be $\frac{2\pi}{N}$. As we increase N to infinity, Ω_0 gets smaller, and the spacing between the coefficients in terms of rad/sec gets smaller and smaller (but the overall shape remains similar). $N = 5$:



Toward the DTFT

If we let $N \to \infty$, then $x_p[\cdot] \to x[\cdot]$, but $x_p[\cdot]$ is still periodic, so we can still represent it with a Fourier series!

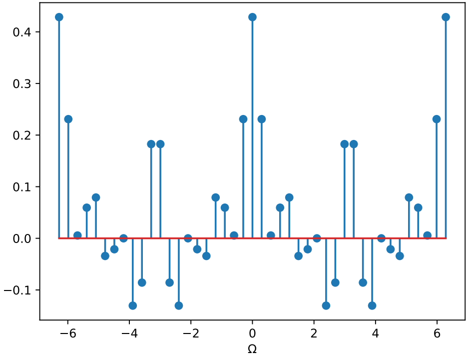
The frequency Ω_0 associated with $k = 1$ is defined to be $\frac{2\pi}{N}$. As we increase N to infinity, Ω_0 gets smaller, and the spacing between the coefficients in terms of rad/sec gets smaller and smaller (but the overall shape remains similar). $N = 11$:



Toward the DTFT

If we let $N \to \infty$, then $x_p[\cdot] \to x[\cdot]$, but $x_p[\cdot]$ is still periodic, so we can still represent it with a Fourier series!

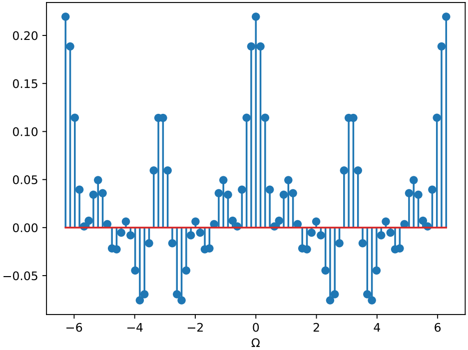
The frequency Ω_0 associated with $k = 1$ is defined to be $\frac{2\pi}{N}$. As we increase N to infinity, Ω_0 gets smaller, and the spacing between the coefficients in terms of rad/sec gets smaller and smaller (but the overall shape remains similar). $N = 21$:



Toward the DTFT

If we let $N \to \infty$, then $x_p[\cdot] \to x[\cdot]$, but $x_p[\cdot]$ is still periodic, so we can still represent it with a Fourier series!

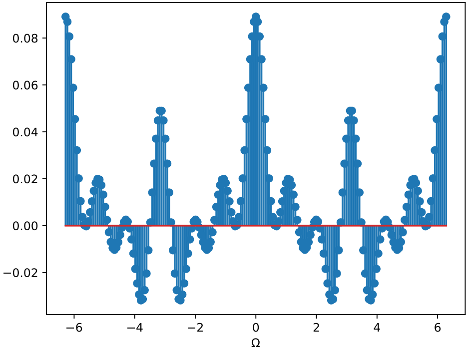
The frequency Ω_0 associated with $k = 1$ is defined to be $\frac{2\pi}{N}$. As we increase N to infinity, Ω_0 gets smaller, and the spacing between the coefficients in terms of rad/sec gets smaller and smaller (but the overall shape remains similar). $N = 41$:



Toward the DTFT

If we let $N \to \infty$, then $x_p[\cdot] \to x[\cdot]$, but $x_p[\cdot]$ is still periodic, so we can still represent it with a Fourier series!

The frequency Ω_0 associated with $k = 1$ is defined to be $\frac{2\pi}{N}$. As we increase N to infinity, Ω_0 gets smaller, and the spacing between the coefficients in terms of rad/sec gets smaller and smaller (but the overall shape remains similar). $N = 101$:



Fourier Series to Fourier Transform

Once we have a periodic signal, we can find the FSC:

$$X_p[k] = \frac{1}{N} \sum_{<N>} x_p[n] e^{-jk\Omega_0 n}$$

where $\Omega_0 = \frac{2\pi}{N}$.

Now we want to think about $N \to \infty$ Let's replace $\frac{1}{N}$ with $\frac{\Omega_0}{2\pi}$, and explicitly pick a period to sum over:

$$X_p[k] = \frac{\Omega_0}{2\pi} \int_{-N/2}^{N/2} x_p[n] e^{-jk\Omega_0 t}$$

Fourier Series to Fourier Transform

Now, substitute into the synthesis equation:

$$x_p[n] = \sum_{k=-N/2}^{N/2} X_p[k] e^{jk\Omega_0 n}$$

$$= \sum_{k=--N/2}^{N/2} \left\{ \frac{\Omega_0}{2\pi} \int_{-N/2}^{N/2} x_p[n] e^{-jk\Omega_0 t} \right\} e^{jk\Omega_0 n}$$

As we take $T \to \infty$, a few things happen:

- $x_p[n] \to x[n]$
- Ω_0 becomes an infinitesimally small value, $\Omega_0 \to \mathrm{d}\Omega$
- $k\Omega_0$ becomes a continuum, $k\Omega_0 \to \Omega$ (continuous)
- The bounds of summation approach $-\infty$ and ∞ (respectively)
- The outer sum becomes an integral.

Discrete-Time Fourier Transform

Synthesis Equation

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega$$

Analysis Equation

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

Discrete-Time Fourier Transform

Synthesis Equation

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega$$

Analysis Equation

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

Discrete-time Fourier Transform

Problem: Find the Fourier transform of the following signal.

$$x[n] = a^n u[n] \quad \text{where} \quad u[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Sketch its magnitude and phase.

Inverse Discrete-Time Fourier Transform

Find the signal whose Fourier transform is

$$X(\Omega) = e^{-j3\Omega}$$

Discrete-Time Fourier Transform

Find the Fourier transforms of the following discrete-time signals.

- $x_1[n] = a^n u[n]$ where $u[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$

- $x_2[n] = a^{(n-n_0)} u[n - n_0]$

- $x_3[n] = \mathrm{Sym}\{a^n u[n]\}$

- $x_4[n] = \mathrm{Asym}\{a^n u[n]\}$

- $x_5[n] = n a^n u[n]$

Discrete-Time Fourier Transform

Find the Fourier transform of
$x_2[n] = a^{(n-n_0)}u[n-n_0]$

Discrete-Time Fourier Transform

Find the Fourier transform of

$x_3[n] = \mathrm{Sym}\{a^n u[n]\}$

Discrete-Time Fourier Transform

Find the Fourier transform of

$x_4[n] = \mathrm{Asym}\{a^n u[n]\}$

Discrete-Time Fourier Transform

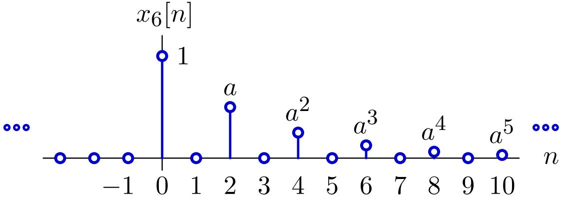
Find the Fourier transform of

$x_5[n] = na^n u[n]$

Discrete-Time Fourier Transform

Find the Fourier transform of $x_6[n]$:

$$x_6[n] = \begin{cases} (a)^{n/2} & n = 0, 2, 4, 6, 8, ..., \infty \\ 0 & \text{otherwise} \end{cases}$$



Plot the magnitude and angle of $X_6(\Omega)$ versus Ω.

Discrete-Time Fourier Transform

$$x_6[n] = \begin{cases} (a)^{n/2} & n = 0, 2, 4, 6, 8, ..., \infty \\ 0 & \text{otherwise} \end{cases}$$

Power Square

Find the sum of the numbers in the infinite quadrant shown below, where $a < 1$.

$$\begin{array}{|cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
a^4 & a^5 & a^6 & a^7 & a^8 & \cdots \\
a^3 & a^4 & a^5 & a^6 & a^7 & \cdots \\
a^2 & a^3 & a^4 & a^5 & a^6 & \cdots \\
a^1 & a^2 & a^3 & a^4 & a^5 & \cdots \\
a^0 & a^1 & a^2 & a^3 & a^4 & \cdots \\
\hline
\end{array}$$