6.003: Signal Processing

we will start @
2:05pm Eastern

Discrete-Time Fourier Series

$x(t)$

Synthesis Equation

$$x[n] = x[n+N] = \sum_{k=\langle N\rangle} X[k]\, e^{j\frac{2\pi k}{N}n}$$

Analysis Equation

$$X[k] = X[k+N] = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-j\frac{2\pi k}{N}n}$$

0, ..., N−1
N, ..., 2N−1

$X[\cdot]$

$X[0]$

4 March 2021

Discrete-Time Fourier Series

Compare the pitches of two signals:

f_1 (Hz) =

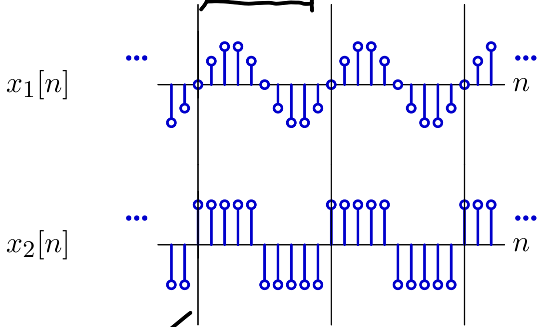
$$x_1[n] = \sin(2\pi n/10)$$

f_2 (Hz) =

$$x_2[n] = \begin{cases} 1 & \text{if } \mathrm{mod}(n,10) < 5 \\ -1 & \text{otherwise} \end{cases}$$

each played at 4 kHz sample rate.

10 samples



$N \frac{\text{samples}}{\text{period}}$

$$f \frac{\text{periods}}{\text{second}} = f_s \frac{\cancel{\text{samples}}}{\text{second}} \cdot \frac{1}{N} \frac{\text{periods}}{\cancel{\text{sample}}} = \frac{f_s}{N} = 400\,\text{Hz}$$

Discrete-Time Fourier Series

Each of the signals in the previous slide is DT.

We can think of each cycle as composed of 10 samples.

When "played," the DT waveform (say 400 cycles) is converted to CT by the hardware in your computer/speakers.

Discrete-Time Fourier Series

Listen to the sounds. Different sounds, same pitch.

Interest in the question of pitch has been longstanding (from music).

Musical instruments are typically described as having three properties: loudness, pitch, and timbre (pronounced tam-ber).

Loudness is easy to understand intuitively.

Pitch has to do with the auditory position of a sound in the space of sounds ("higher" or "lower" pitches)

Timbre is more difficult to define. It's often defined as the "quality" of a sound, as distinct from its loudness and pitch.

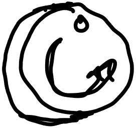
Why do the sounds on the previous slide have the same pitch?
What does that mean physically?

Pitch Experiments

Early experiments in the field of psychoacoustics were limited by their ability to manipulate sounds.

For example, musical instruments based on strings or columns of air were known to exhibit a (more or less) characteristic harmonic pattern.

Not clear how to think about harmonics. Do they contribute to pitch? timbre? loudness?

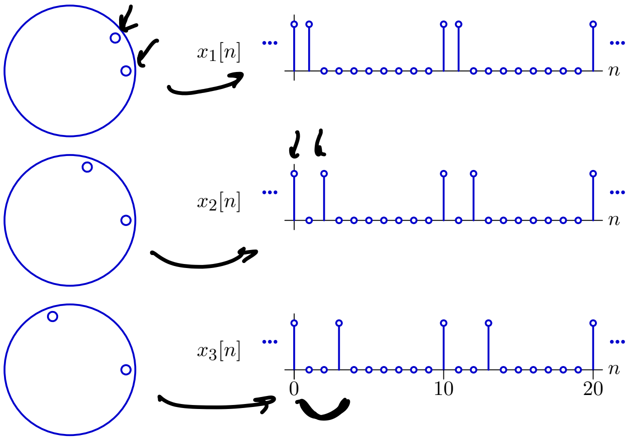


Later, Seebeck used sirens to generate more complicated sounds.

Interesting experiments with controversial results.

Sirens

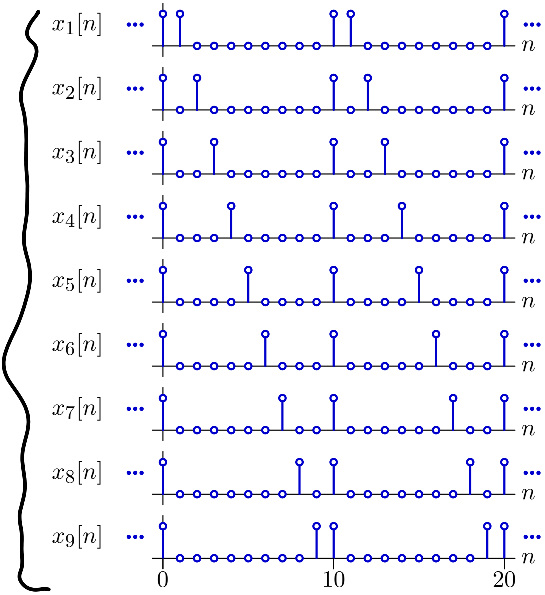
Seebeck used a siren to generate more complicated sounds (circa 1841). Sounds were made by passing a jet of compressed air through holes in a spinning disk.



The pattern of holes determined the pattern of pulses in each period.
The speed of spinning controlled the number of periods per second.

Interpreting Complex Sounds

Seebeck found interesting phenomena based on periodicity of holes.



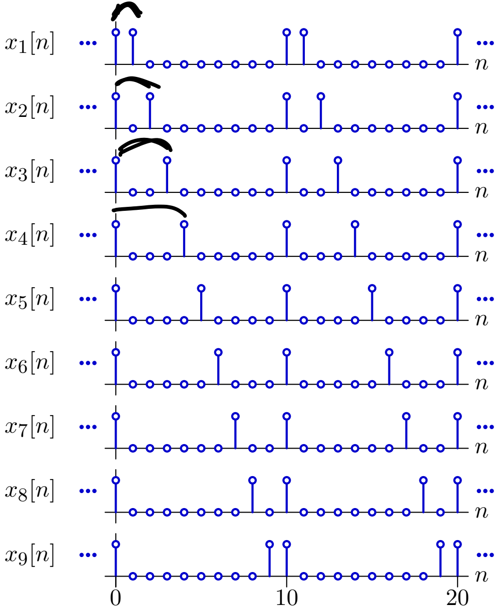
Listen to these signals played with 4 kHz sample rate.

Fourier Interpretation

Find Fourier series representations of Seebeck's signals.

$x_m[n]$

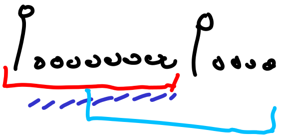
$x[n] + x[n-m]$



$x_{11}[n$

Fourier Interpretation

Start with $x[n] = x[n+10] = \begin{cases} 1 & \text{if } n \bmod 10 \equiv 0 \\ 0 & \text{otherwise} \end{cases}$



$$X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j2\pi k n / N}$$

$N=10$

$$= \frac{1}{N} \sum_{n=0}^{9} x[n] e^{-j2\pi k n / N} = \frac{1}{10} \cancel{x[0]} e^{\cancel{-j2\pi k(0)/10}}$$

$$= \frac{1}{10}, \quad \forall k$$

$a + bj$

real imag

How to find $X_m[k]$ in terms of $X[k]$?

$$x_m[n] = x[n] + x[n-m]$$

$$X_m[k] = \frac{1}{10}\left(1 + e^{-j2\pi k m/10}\right)$$

$$\frac{1}{10} \sum_{n=5}^{14} x[n] e^{-j2\pi k n / N}$$

$$\frac{1}{10} \cancel{x[10]} e^{\cancel{-j2\pi k}}$$

$$x_c[n] = \boxed{A x_a[n] + B x_b[n]} \quad \text{linearity}$$

$$\downarrow \text{DTFS}, N \qquad \downarrow \text{DTFS}, N$$

$$\underline{X_c[k]} = A \underline{X_a[k]} + B \underline{X_b[k]}$$

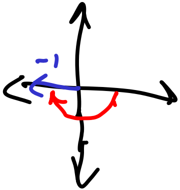
Fourier Interpretation

Find Fourier series representations of Seebeck's signals.

$$X_m[k] = \frac{1}{10}\sum_{n=0}^{9} x_m[n]\, e^{-j\frac{2\pi k}{10}n} = \frac{1}{10}\left(1 + e^{-j\frac{2\pi k}{10}m}\right)$$

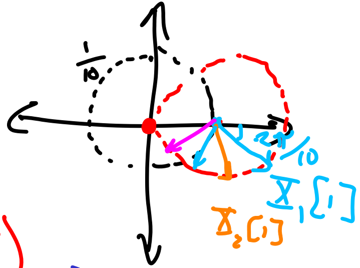
DC: $k = 0$ term

$$X_m[0] = \frac{2}{10}$$



Fundamental: $k = 1$ term

$$X_m[1] = \frac{1}{10}\left(1 + e^{-j\frac{2\pi m}{10}}\right)$$



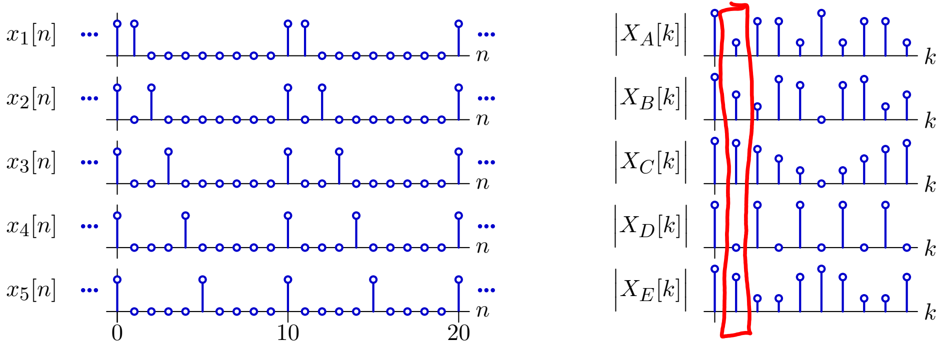
$$X_5[1] = \frac{1}{10}\left(1 + e^{-j\frac{2\pi(5)}{10}}\right) = \frac{1}{10}\left(1 + e^{-j\pi}\right) = 0$$

-1

Fourier Matching

Fundamental: $k = 1$ term

Which series corresponds to each time function on the left (below)?

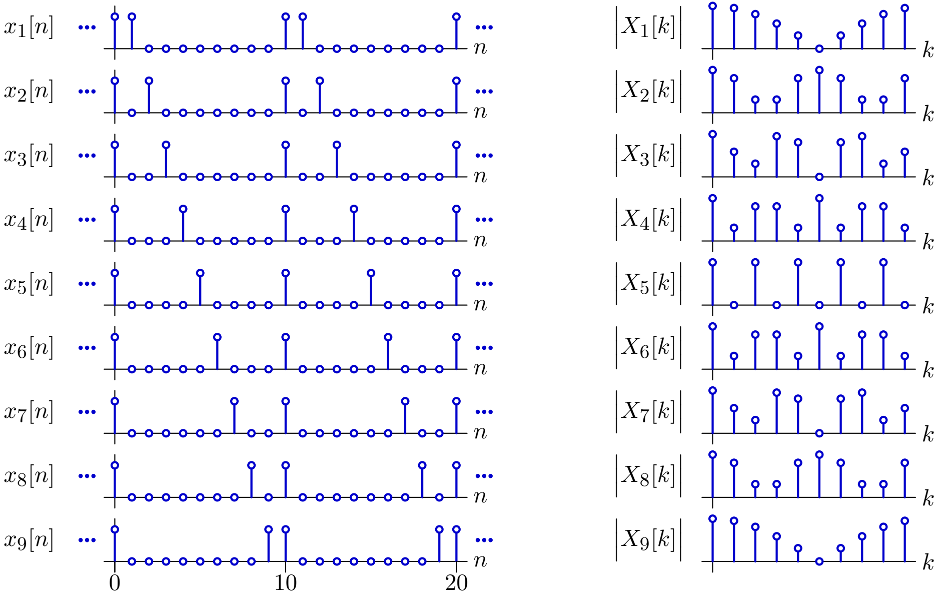


Multiple ways to see this, but one way is to look at $|X_m[1]|$, highlighted above. From the previous slide, should be biggest for $|X_1[1]|$, smallest for $|X_5[1]|$, monotonically decreasing with m.

Fourier Series

Why are the magnitude functions for $x_2[\cdot]$ and $x_8[\cdot]$ the same?

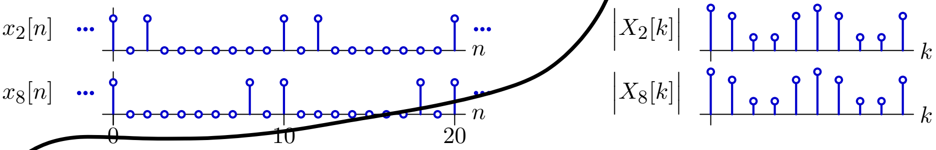
Are the series coefficients the same?



Fourier Series

Why are the magnitude functions for $x_2[\cdot]$ and $x_8[\cdot]$ the same?

Are the series coefficients the same? No!



$x_2[n] = x_8[n-2]$

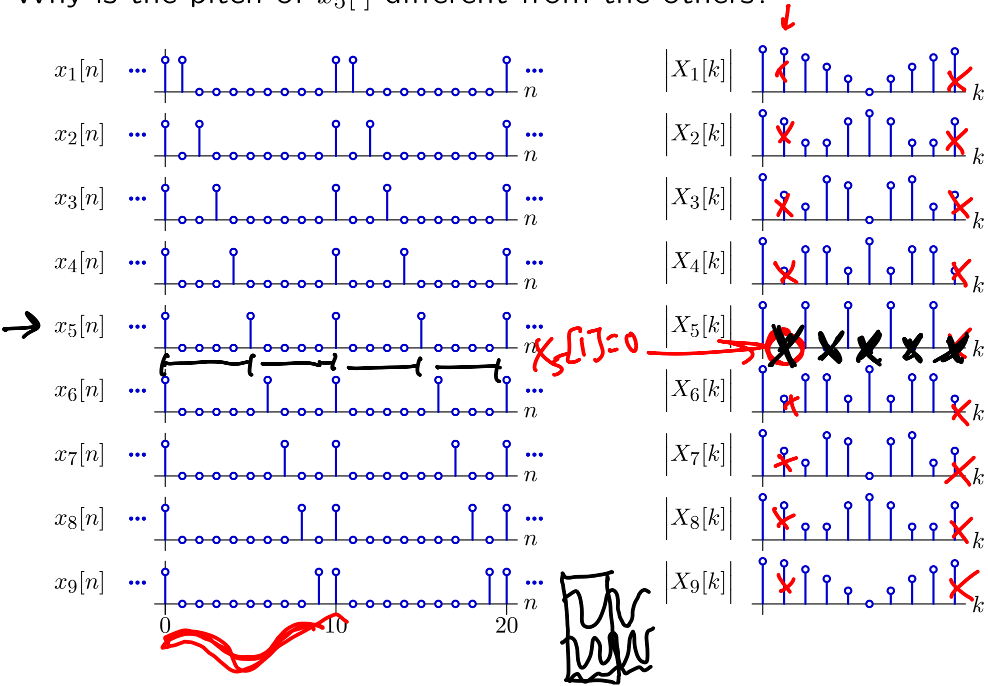
differ only by shift in time → differ by phase in frequency

$$X_2[k] = X_8[k]\, e^{-j\frac{4\pi k}{10}}$$

→ same magnitude as $X_8[k]$, but different angle!

Fourier Series

Why is the pitch of $x_5[\cdot]$ different from the others?



Missing Fundamental

An important feature of $x_5[\cdot]$ is that it has no fundamental component (when analyzed with $N = 10$). Could this missing fundamental explain the difference in percept?

We can test this hypothesis by removing the fundamental component from each of Seebeck's waveforms.

How would you create signals $y_i[\cdot]$ that have the same harmonics as $x_i[\cdot]$ but no energy at the fundamental frequency? (nofund.wav)

$$x_m[n] = \sum_{k=0}^{9} X[k] e^{j\frac{2\pi k n}{10}}$$

$$x_m[n] \rightarrow X_m[k] \rightarrow \boxed{\begin{array}{l} X_m[1] = 0 \\ X_m[-1] = 0 \end{array}} \rightarrow \text{resynthesize}$$

need both!

Fourier Series of $X_i[\cdot]$ and $Y_i[\cdot]$

Listen to the signals after the fundamentals have been removed.