Complex Numbers

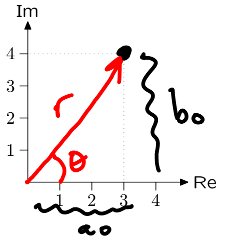
Complex numbers consist of real and imaginary parts. You may already be familiar with complex numbers written in their *rectangular form*:

$$\boxed{a_0 + b_0 j}$$

where $j = \sqrt{-1}$. Here, a_0 is called the *real part* and b_0 is called the *imaginary part*.

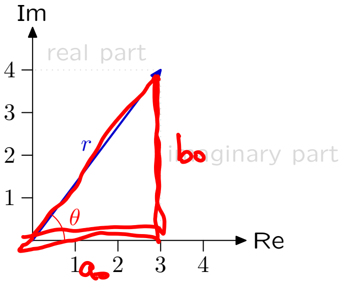
We will often represent these numbers using a 2-d space we'll call the *complex plane*. For example, here is the number $3 + 4j$ represented as a point:

complex plane



Complex Numbers: Polar Form

Importantly, we can also think of these numbers as *vectors* in the complex plane. For example, here is $3 + 4j$ represented as a vector:



We have labeled this number with a magnitude (r) and an angle (θ). Let's also note that we can define relationships between a_0, b_0, r, and θ:

$$r = \sqrt{a_0^2 + b_0^2} \qquad \theta = \tan^{-1}\left(\frac{b_0}{a_0}\right)$$

$$a_0 = r\cos\theta \qquad b_0 = r\sin\theta$$

Complex Numbers: Polar Form

a_0 b_0

From there, we can rewrite $a_0 + b_0 j$ as: $r\,(\cos(\theta) + j\sin(\theta))$.

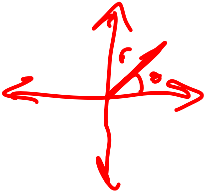
Then we can use Euler's equation ($e^{jx} = \cos(x) + j\sin(x)$) to express our complex number as:

$$\boxed{re^{j\theta}}$$

polar form

This representation of complex numbers is known as the *polar form*. Here, r is called the *magnitude* of the number, and θ is called the *phase* (or *angle*, or *argument*) of the number.

$$\boxed{}\,e^{j\boxed{}}$$



Operations on Complex Numbers

Addition/Subtraction: real and imaginary parts sum independently

$$c_1 + c_2 = \underbrace{a_1 + b_1 j} + \underbrace{a_2 + b_2 j} = (a_1 + a_2) + (b_1 + b_2)j$$

rectangular form is nice for addition/subtraction.

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Multiplication/Division: complicated in rectangular form!

$$c_1 c_2 = (a_1 + b_1 j)(a_2 + b_2 j) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) j$$

GROSS!

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GROSS!

Polar form is much nicer: magnitudes multiply, angles add

$$c_1 c_2 = (r_1 e^{j\theta_1})(r_2 e^{j\theta_2}) = (r_1 r_2) e^{j(\theta_1 + \theta_2)}$$

Complex Exponentials

From Euler's formula, we have several useful ways of converting between trig functions and complex exponentials:

$$e^{jx} = \cos(x) + j\sin(x)$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} = -\frac{j}{2}\left(e^{jx} - e^{-jx}\right)$$

Complex Numbers

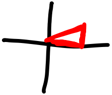
How many of the following are true?

- yes $\dfrac{1}{\cos\theta + j\sin\theta} \overset{?}{=} \cos\theta - j\sin\theta$ ✓

  $\dfrac{1}{e^{j\theta}} = e^{-j\theta} = \cos(-\theta) + j\sin(-\theta)$, $e^{j(-\theta)}$

- yes $(\cos\theta + j\sin\theta)^n \overset{?}{=} \cos(n\theta) + j\sin(n\theta)$ ✓

  $(e^{j\theta})^n = e^{j(n\theta)}$

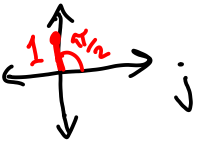


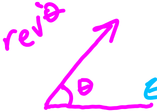
- no $\mathrm{Im}\left(j^j\right) \overset{?}{>} \mathrm{Re}\left(j^j\right)$ ✗

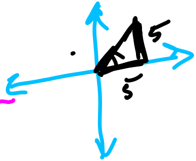
- yes $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) \overset{?}{=} \tan^{-1} 1$ ✓

j^j $\quad \boxed{j}$

$\left(e^{j\frac{\pi}{2}}\right)^j = e^{-\frac{\pi}{2}}$







$c_1 = 2 + j \dashrightarrow \angle c_1 = \tan^{-1}\left(\frac{1}{2}\right)$

$c_2 = 3 + j \dashrightarrow \angle c_2 = \tan^{-1}\left(\frac{1}{3}\right)$

$c_1 \cdot c_2 \dashrightarrow \angle(c_1 \cdot c_2)$

$= \angle c_1 + \angle c_2$

$\downarrow$

$5 + 5j \qquad = \tan^{-1}(1)$

Continuous-Time Fourier Series

Complex exponential form.

Synthesis Equation (making a signal from components):

$$\underline{x(t) = x(t+T)} = \sum_{k=-\infty}^{\infty} X[k]\, e^{j\frac{2\pi k}{T}t}$$

Analysis Equation (finding the components):

$$\underline{X[k]} = \frac{1}{T}\int_T x(t)\, \underline{e^{-j\frac{2\pi k}{T}t}}\, dt$$

$c_0 = \int$

$c_k = \int$

$d_k = \int$

Warm Up

Find the Fourier series components $X[k]$ for

$$x(t) = \cos(t) = \frac{1}{2}\left(e^{jt} + e^{-jt}\right)$$

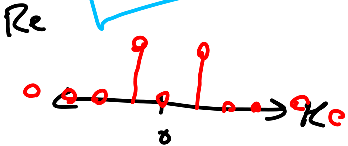
$T = 2\pi$

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]\, e^{j\frac{2\pi k}{T}t} = \sum_{k=-\infty}^{\infty} X[k]\, e^{jkt}$$

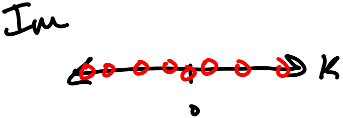
real, symmetric $f(t)$

real, symmetric $F[k]$

$$X[k] = \frac{1}{T}\int_T x(t)\, e^{-j\frac{2\pi k}{T}t}\, dt$$



$$X[k] = \begin{cases} \frac{1}{2}, & |k| = 1 \\ 0, & \text{otherwise} \end{cases}$$



Warm Up

Find the Fourier series components $X[k]$ for

$$x(t) = \sin(t) = \frac{-j}{2}\left(e^{jt} - e^{-jt}\right)$$

$T = 2\pi$

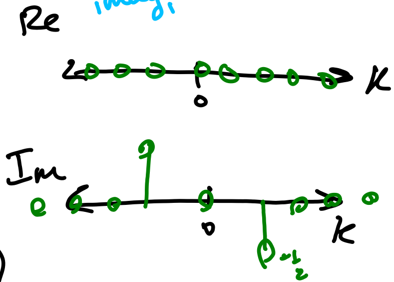
$$x(t) = \sum_{k=-\infty}^{\infty} X[k]\, e^{j\frac{2\pi k}{T}t} = \sum_{k=-\infty}^{\infty} X[k]\, e^{jkt}$$

synthesis

real, antisym $\quad f(t)$

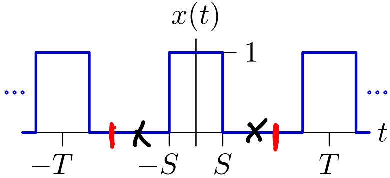
imag, antisym $\quad F[k]$

$$X[k] = \begin{cases} -j/2, & k = 1 \\ j/2, & k = -1 \\ 0, & \text{o.w.} \end{cases}$$



Pulse Train

Find the Fourier series coefficients $X[k]$ for $x(t)$:



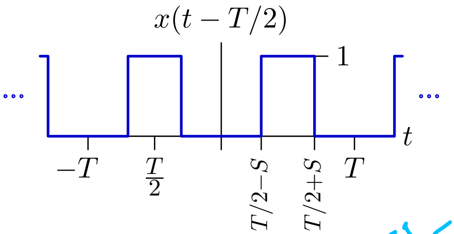
$$X[k] = \frac{1}{T}\int_T x(t)\, e^{-j\frac{2\pi k t}{T}}\, dt$$

$$= \frac{1}{T}\int_{-S}^{S} (1)\, e^{-j\frac{2\pi k t}{T}}\, dt = \frac{T}{-j2\pi k}\,\frac{1}{T}\left(e^{-j\frac{2\pi k t}{T}}\right)\Bigg|_{t=-S}^{S}$$

$$= \frac{-1}{j2\pi k}\left(e^{-j\frac{2\pi k S}{T}} - e^{j\frac{2\pi k S}{T}}\right) = \frac{\sin\left(\frac{2\pi k S}{T}\right)}{\pi k}$$

Pulse Train

What would happen to Fourier series if you delayed $x(t)$ by $T/2$?



In fact, what about an arbitrary shift t_0?

$$y(t) = x(t-t_0)$$

$$Y[k] = \frac{1}{T}\int_T x(t-t_0)\, e^{-j\frac{2\pi k t}{T}}\, dt$$

let $m = t - t_0$

$$= \frac{1}{T}\int_T x(m)\, e^{-j\frac{2\pi k(m+t_0)}{T}}\, dm = \frac{1}{T}\int_T x(m)\, e^{-j\frac{2\pi k m}{T}}\, e^{-j\frac{2\pi k t_0}{T}}\, dm$$

$$= e^{-j\frac{2\pi k t_0}{T}}\left(\frac{1}{T}\int_T x(m)\, e^{-j\frac{2\pi k m}{T}}\, dm\right)$$

$$Y[k] = e^{-j\frac{2\pi k t_0}{T}}\, \underline{X[k]}$$

Delay Property of Fourier Series

Complex exponential form simplifies expression of delay property.

yay

delay	complex exponential form	trig form
$T/2$	mult by $e^{-j\pi k}$	$c_k = \begin{cases} c_k, & k \text{ even} \\ -c_k, & k \text{ odd} \end{cases}$
$T/4$	mult by $e^{-j\pi k/2}$	$c'_k = d_k$ $d'_k = -c_k$
t_0	mult by $e^{-j\frac{2\pi k t_0}{T}}$	COMPLICATED

Fourier Series Matching

Match the signals (left column) to Fourier series coefficients (right).