6.003: Signal Processing

Continuous-Time Fourier Series
(Complex Exponential Form)

25 February 2021

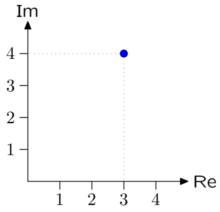
Complex Numbers

Complex numbers consist of real and imaginary parts. You may already be familiar with complex numbers written in their *rectangular form*:

$$a_0 + b_0 j$$

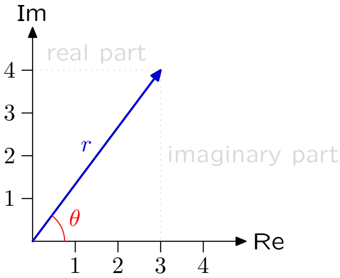
where $j = \sqrt{-1}$. Here, a_0 is called the *real part* and b_0 is called the *imaginary part*.

We will often represent these numbers using a 2-d space we'll call the *complex plane*. For example, here is the number $3 + 4j$ represented as a point:



Complex Numbers: Polar Form

Importantly, we can also think of these numbers as *vectors* in the complex plane. For example, here is $3+4j$ represented as a vector:



We have labeled this number with a magnitude (r) and an angle (θ). Let's also note that we can define relationships between a_0, b_0, r, and θ:

$$r = \sqrt{a_0^2 + b_0^2} \qquad \theta = \tan^{-1}\left(\frac{b_0}{a_0}\right)$$

$$a_0 = r\cos\theta \qquad b_0 = r\sin\theta$$

Complex Numbers: Polar Form

From there, we can rewrite $a_0 + b_0 j$ as: $r\left(\cos(\theta) + j\sin(\theta)\right)$.

Then we can use Euler's equation ($e^{jx} = \cos(x) + j\sin(x)$) to express our complex number as:

$$re^{j\theta}$$

This representation of complex numbers is known as the *polar form*. Here, r is called the *magnitude* of the number, and θ is called the *phase* (or *angle*, or *argument*) of the number.

Operations on Complex Numbers

Addition/Subtraction: real and imaginary parts sum independently

$$c_1 + c_2 = a_1 + b_1 j + a_2 + b_2 j = (a_1 + a_2) + (b_1 + b_2) j$$

rectangular form is nice for addition/subtraction.

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Multiplication/Division: complicated in rectangular form!

$$c_1 c_2 = (a_1 + b_1 j)(a_2 + b_2 j) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) j$$

GROSS!

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GROSS!

Polar form is much nicer: magnitudes multiply, angles add

$$c_1 c_2 = (r_1 e^{j\theta_1})(r_2 e^{j\theta_2}) = (r_1 r_2) e^{j(\theta_1 + \theta_2)}$$

Complex Exponentials

From Euler's formula, we have several useful ways of converting between trig functions and complex exponentials:

$$e^{jx} = \cos(x) + j\sin(x)$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} = -\frac{j}{2}\left(e^{jx} - e^{-jx}\right)$$

Complex Numbers

How many of the following are true?

- $$\frac{1}{\cos\theta + j\sin\theta} = \cos\theta - j\sin\theta$$
- $$(\cos\theta + j\sin\theta)^n = \cos(n\theta) + j\sin(n\theta)$$
- $$\operatorname{Im}\left(j^j\right) > \operatorname{Re}\left(j^j\right)$$
- $$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1} 1$$

Continuous-Time Fourier Series

Complex exponential form.

Synthesis Equation (making a signal from components):

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} X[k]\, e^{j\frac{2\pi k}{T}t}$$

Analysis Equation (finding the components):

$$X[k] = \frac{1}{T}\int_T x(t)\, e^{-j\frac{2\pi k}{T}t}\, dt$$

Warm Up

Find the Fourier series components $X[k]$ for

$$x(t) = \cos(t)$$

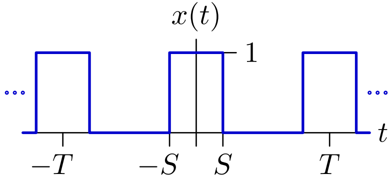
Warm Up

Find the Fourier series components $X[k]$ for

$$x(t) = \sin(t)$$

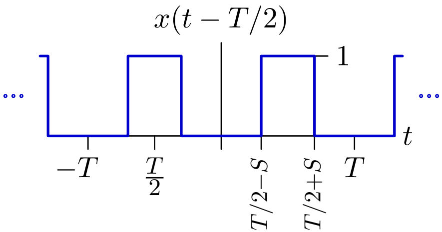
Pulse Train

Find the Fourier series coefficients $X[k]$ for $x(t)$:



Pulse Train

What would happen to Fourier series if you delayed $x(t)$ by $T/2$?



In fact, what about an arbitrary shift t_0?

Delay Property of Fourier Series

Complex exponential form simplifies expression of delay property.

delay	complex exponential form	trig form
$T/2$		
$T/4$		
t_0		

Fourier Series Matching

Match the signals (left column) to Fourier series coefficients (right).