6.003: Signal Processing

Continuous-Time Fourier Series (Trig Form)

good morning!

we will start @ 8:05am Eastern

23 February 2021

Fourier Representations

Key Idea: Assume that we can write an arbitrary periodic function $f(t) = f(t+T)$ as a sum of harmonically-related sinusoids periodic in T:

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(\frac{2\pi kt}{T}\right) + \sum_{k=1}^{\infty} d_k \sin\left(\frac{2\pi kt}{T}\right)$$

We have already seen an example of this process in lab 1! Combining harmonically-related sines and cosines using the given coefficients produced a very complicated signal (human speech).

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Today, we'll focus on the reverse problem: *given* a signal that can be represented in this way, how can we find the coefficients?

Preliminaries: Sinusoids

Summing two sinusoids with the same frequency yields a (scaled, shifted) sinusoid at that same frequency.

$$A\cos(\omega t+\phi_1)+B\cos(\omega t+\phi_2)=C\cos(\omega t+\phi_3)$$

where

$$C=\sqrt{(A\cos(\phi_1)+B\cos(\phi_2))^2+(A\sin(\phi_1)+B\sin(\phi_2))^2}$$

and

$$\phi_3=\tan^{-1}\left(\frac{A\sin(\phi_1)+B\sin(\phi_2)}{A\cos(\phi_1)+B\cos(\phi_2)}\right)$$

It follows that:

$$A\cos(\omega t)+B\sin(\omega t)=\left(\sqrt{A^2+B^2}\right)\cos\left(\omega t+\tan^{-1}\left(\frac{-B}{A}\right)\right)$$

Preliminaries: Sinusoids

Integrating a sinusoid over a full period yields 0. Where k is a positive integer:

$$\int_{t_0}^{t_0+T} \sin\left(\frac{2\pi k}{T} t\right) \mathrm{d}t = 0$$

$$\int_{t_0}^{t_0+T} \cos\left(\frac{2\pi k}{T} t\right) \mathrm{d}t = 0$$

Preliminaries: Sinusoids

Harmonically-related sines and cosines are *orthogonal*. Where k and m are positive integers:

$$\int_{t_0}^{t_0+T} \sin\left(\frac{2\pi k}{T}t\right)\cos\left(\frac{2\pi m}{T}t\right)\mathrm{d}t = 0$$

Preliminaries: Sinusoids

Where k and m are positive integers:

$$\int_{t_0}^{t_0+T} \cos\left(\frac{2\pi k}{T}t\right)\cos\left(\frac{2\pi m}{T}t\right)\mathrm{d}t = \boxed{\begin{cases} T/2 & \text{if } k = m, \\ 0 & \text{otherwise} \end{cases}}$$

Preliminaries: Sinusoids

Where k and m are positive integers:

$$\int_{t_0}^{t_0+T} \sin\left(\frac{2\pi k}{T}t\right)\sin\left(\frac{2\pi m}{T}t\right)\mathrm{d}t = \begin{cases} T/2 & \text{if } k = m, \\ 0 & \text{otherwise} \end{cases}$$

Finding Coefficients

Given these properties of sinusoids, and given a function:

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(\frac{2\pi k t}{T}\right) + \sum_{k=1}^{\infty} d_k \sin\left(\frac{2\pi k t}{T}\right)$$

how can we find the coefficients c_k and d_k?

c_0: The "DC" Term

For largely historical reasons, the constant c_0 is referred to as the "DC" term: it represents the constant offset of a signal (if any). Stated another way, it represents the *average value* of a signal over one period:

$$c_0 = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\mathrm{d}t$$

Computing the Other c_k Terms

We are assuming that our signal is comprised of harmonically-related sines and cosines. Here is an example signal:

$$\int f(t) = \int \cancel{c_1 \cos(\omega_0 t)} + \int \cancel{d_1 \sin(\omega_0 t)} + \int c_3 \cos(3\omega_0 t) + \int \cancel{d_5 \sin(5\omega_0 t)}$$

$\cos(3\omega_0 t)$ $\quad 0 \quad 0 \quad \cos(3\omega_0 t) \quad 0$

Let's imagine we wanted to determine the coefficient c_3, associated with $\cos(3\omega_0 t)$.

$= \frac{T}{2} c_3$

Our strategy will be to multiply $f(t)$ by $\cos(3\omega_0 t)$ and integrate the result over one period.

Check Yourself: How does this help us find c_3?

Computing the Other Terms

In general, we can compute the coefficients:

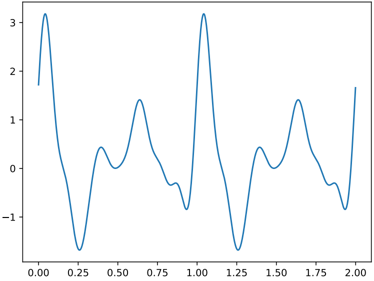
$$c_k = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos\left(\frac{2\pi kt}{T}\right)\mathrm{d}t = \frac{2}{T}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} f(t)\cos(k\omega_0 t)\mathrm{d}t$$

We can use similar logic to find the coefficients associated with the sine waves:

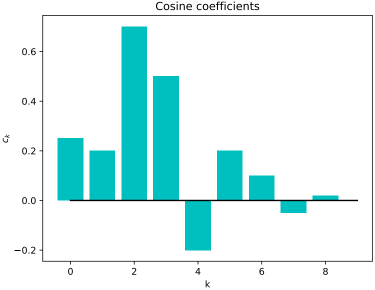
$$d_k = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin\left(\frac{2\pi kt}{T}\right)\mathrm{d}t = \frac{2}{T}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} f(t)\sin(k\omega_0 t)\mathrm{d}t$$

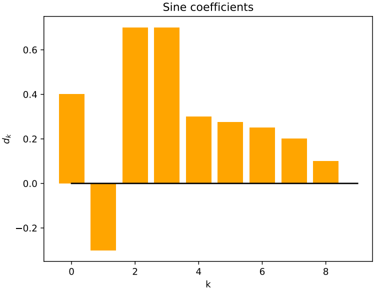
Two Different Views

Time domain:



Frequency domain:





Continuous-Time Fourier Series (Trig Form)

Synthesis Equation (making a signal from components):

$$\underline{f(t) = f(t+T)} = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(\frac{2\pi k t}{T}\right) + \sum_{k=1}^{\infty} d_k \sin\left(\frac{2\pi k t}{T}\right)$$

Analysis Equations (finding the components):

$$c_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,\mathrm{d}t$$

$$c_k = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos\left(\frac{2\pi k t}{T}\right)\,\mathrm{d}t \; ; \quad k > 0$$

$$d_k = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin\left(\frac{2\pi k t}{T}\right)\,\mathrm{d}t$$

Warm Up

Find the Fourier series components (c_k and d_k) for

$$x(t) = x(t+2\pi) = \boxed{\cos(t)}$$

$T = 2\pi$ seconds

match up

$$x(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(kt) + \sum_{k=1}^{\infty} d_k \sin(kt)$$

$d_k = 0$

$c_0 = 0$

$$c_k = \begin{cases} 1, & k=1 \\ 0, & \text{otherwise} \end{cases}$$

or, directly using analysis if you want to:

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} \cos(t)\,dt = 0$$

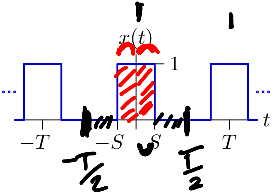
$$c_k = \frac{2}{2\pi}\int_0^{2\pi} \cos(t)\cos(kt)\,dt = \begin{cases} \frac{2}{2\pi}\frac{2\pi}{2} = 1, & \text{if } k=1 \\ 0 & \text{otherwise} \end{cases}$$

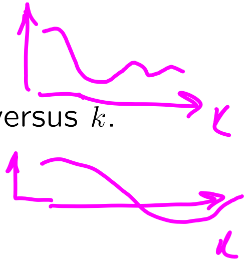
$$d_k = \frac{2}{2\pi}\int_0^{2\pi} \cos(t)\sin(kt)\,dt = 0$$

Pulse Train

Find the Fourier series coefficients (c_k and d_k) for

0 because of symmetry





Find the Fourier series coefficients and plot them versus k.

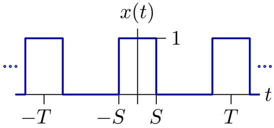
$$c_0 = \frac{1}{T}\int_{\langle T\rangle} x(t)\,dt = \frac{1}{T}\int_{-S}^{S} dt = \frac{2S}{T}$$

$$c_k = \frac{2}{T}\int_{\langle T\rangle} x(t)\cos\left(\frac{2\pi k}{T}t\right)dt = \frac{2}{T}\int_{-S}^{S}\cos\left(\frac{2\pi k}{T}t\right)dt$$

$$= \frac{4}{T}\int_{0}^{S}\cos\left(\frac{2\pi k}{T}t\right)dt = \frac{\cancel{4}^{\,2}}{\cancel{T}}\,\frac{\cancel{T}}{\cancel{2}\pi k}\sin\left(\frac{2\pi k}{T}t\right)\Big|_{t=0}^{S} = \boxed{\frac{2}{\pi k}\sin\left(\frac{2\pi S k}{T}\right)}$$

Pulse Train

Find the Fourier series coefficients (c_k and d_k) for



What is the effect of scaling time (i.e., scaling S, T proportionally)?

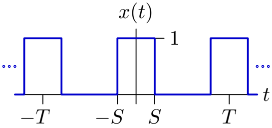
no effect! coefficients depend on $\frac{S}{T}$

coeffs tell us the shape within one period,

T tells us the length of one period

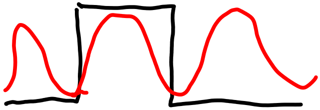
Pulse Train

Find the Fourier series coefficients (c_k and d_k) for

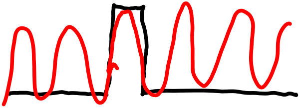


What is effect of increasing only T? … or decreasing only S?

$c_k = \sim \ \sin(\ k\ )$



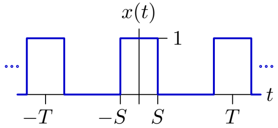
leads to a more "spread out" plot of c_k vs k



$$c_k = \boxed{\frac{2}{\pi k}\sin\left(\frac{2\pi S k}{T}\right)}$$

Pulse Train

Find the Fourier series coefficients (c_k and d_k) for



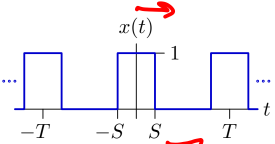
Synthesize with just N terms. What happens as you change N?

we start to approach $x(t)$, but always see overshoot (Gibbs phenomenon)

as N gets bigger, oscillations increase in frequency, and overshoot is more local to the point of discontinuity

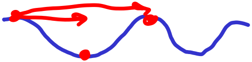
Pulse Train

Find the Fourier series coefficients (c_k and d_k) for



What would happen if you delayed $x(t)$ by $T/2$?

$$x(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(\frac{2\pi k}{T} t\right)$$



$$x\left(t-\frac{T}{2}\right) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(\frac{2\pi k}{T}\left(t-\frac{T}{2}\right)\right)$$

$$= c_0 + \sum_{k=1}^{\infty} c_k \cos\left(\frac{2\pi k}{T} t - \pi k\right)$$

$$= c_0 + \sum_{k=1}^{\infty} \boxed{c_k(-1)^k} \cos\left(\frac{2\pi k}{T} t\right)$$

shift by $\pi \rightarrow$ mult by -1

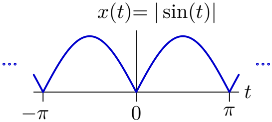
$2\pi \rightarrow$ mult 1

Rectified Sine Wave

Find the Fourier series coefficients (c_k and d_k) for

$$x(t) = x(t+\pi) = |\sin(t)|$$

symmetric $\Rightarrow \forall k,\ d_k = 0$



Find the Fourier series coefficients and plot them versus k.

$$c_0 = \frac{1}{\pi}\int_0^{\pi} \sin(t)\,dt = \frac{1}{\pi}\left(-\cos(t)\right)\Big|_{t=0}^{\pi} = \frac{1}{\pi}\left(1-(-1)\right) = \frac{2}{\pi}$$

$$c_k = \frac{2}{\pi}\int_0^{\pi} \underline{\sin(t)\cos(2kt)}\,dt = \left(\frac{-4}{4k^2-1}\right)\frac{1}{\pi}$$

(full steps on next page)

why is this not 0?

because we're not integrating over a full fundamental period of $\sin(t)$ or $\cos(2kt)$!

$$c_k = \frac{2}{\pi}\int_0^{\pi} \sin(t)\cos(2kt)\,dt$$

$$\Downarrow$$

$$\sin(u)\cos(v) = \frac{1}{2}\left(\sin(u+v) - \sin(v-u)\right)$$

$$\Downarrow$$

$$= \frac{\cancel{2}}{\pi}\frac{1}{\cancel{2}}\left(\int_0^{\pi} \sin((2k+1)t)\,dt - \int_0^{\pi} \sin((2k-1)t)\,dt\right)$$

$$= \frac{1}{\pi}\left(\frac{-1}{2k+1}\cos((2k+1)t)\Big|_{t=0}^{\pi} - \frac{-1}{2k-1}\cos((2k-1)t)\Big|_{t=0}^{\pi}\right)$$

$$\Downarrow$$

$$\cos(m\pi) = \begin{cases} -1 & \text{if } m \text{ odd,} \\ 1 & \text{if } m \text{ even} \end{cases}$$

$$\Downarrow$$

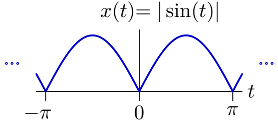
$$= \frac{1}{\pi}\left(\frac{-1}{2k+1}(-1-1) - \frac{-1}{2k-1}(-1-1)\right)$$

$$= \frac{1}{\pi}\left(\frac{2}{2k+1} - \frac{2}{2k-1}\right) = \frac{1}{\pi}\left(\frac{4k-2-4k-2}{4k^2-1}\right) = \frac{1}{\pi}\left(\frac{-4}{4k^2-1}\right)$$

Rectified Sine Wave

Find the Fourier series coefficients (c_k and d_k) for

$$x(t) = x(t + \pi) = |\sin(t)|$$



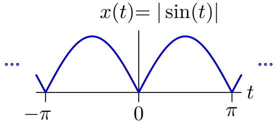
Synthesize with just N terms. Do you see Gibb's phenomenon?

No Gibbs because there is no step discontinuity in $x(t)$!

Rectified Sine Wave

Find the Fourier series coefficients (c_k and d_k) for

$$x(t) = x(t+\pi) = |\sin(t)|$$



Find the Fourier series coefficients for derivative of the rectified sine. Will there be Gibb's?

Yep! $x'(t)$ will be discontinuous at $t = m\pi$, $m \in \mathbb{Z}$

(try plotting partial sums using result from next page!)

(can also expect that c'_k will be 0, since $x'(t)$ will be antisymmetric function of t)

Derivative

Consider:

$$x(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(\frac{2\pi k}{T} t\right) + \sum_{k=1}^{\infty} d_k \sin\left(\frac{2\pi k}{T} t\right)$$

What are the FSC associated with $x'(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}$?

$$x'(t) = \frac{\partial x(t)}{\partial t} = \sum_{k=1}^{\infty} \boxed{\frac{-2\pi k}{T} c_k} \sin\left(\frac{2\pi k}{T} t\right) + \sum_{k=1}^{\infty} \boxed{\frac{2\pi k}{T} d_k} \cos\left(\frac{2\pi k}{T} t\right)$$

$$x'(t) = c'_0 + \sum_{k=1}^{\infty} c'_k \cos\left(\frac{2\pi k}{T} t\right) + \sum_{k=1}^{\infty} d'_k \sin\left(\frac{2\pi k}{T} t\right)$$

$$d'_k = \frac{-2\pi k}{T} c_k$$

$$c'_0 = 0$$

$$c'_k = \frac{2\pi k}{T} d_k$$