6.003: Signal Processing

Continuous-Time Fourier Series (Trig Form)

23 February 2021

Fourier Representations

Key Idea: Assume that we can write an arbitrary periodic function $f(t) = f(t+T)$ as a sum of harmonically-related sinusoids periodic in T:

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(\frac{2\pi kt}{T}\right) + \sum_{k=1}^{\infty} d_k \sin\left(\frac{2\pi kt}{T}\right)$$

We have already seen an example of this process in lab 1! Combining harmonically-related sines and cosines using the given coefficients produced a very complicated signal (human speech).

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Today, we'll focus on the reverse problem: *given* a signal that can be represented in this way, how can we find the coefficients?

Preliminaries: Sinusoids

Summing two sinusoids with the same frequency yields a (scaled, shifted) sinusoid at that same frequency.

$$A\cos(\omega t + \phi_1) + B\cos(\omega t + \phi_2) = C\cos(\omega t + \phi_3)$$

where

$$C = \sqrt{(A\cos(\phi_1) + B\cos(\phi_2))^2 + (A\sin(\phi_1) + B\sin(\phi_2))^2}$$

and

$$\phi_3 = \tan^{-1}\left(\frac{A\sin(\phi_1) + B\sin(\phi_2)}{A\cos(\phi_1) + B\cos(\phi_2)}\right)$$

It follows that:

$$A\cos(\omega t) + B\sin(\omega t) = \left(\sqrt{A^2 + B^2}\right)\cos\left(\omega t + \tan^{-1}\left(\frac{-B}{A}\right)\right)$$

Preliminaries: Sinusoids

Integrating a sinusoid over a full period yields 0. Where k is a positive integer:

$$\int_{t_0}^{t_0+T} \sin\left(\frac{2\pi k}{T} t\right) \mathrm{d}t = 0$$

$$\int_{t_0}^{t_0+T} \cos\left(\frac{2\pi k}{T} t\right) \mathrm{d}t = 0$$

Preliminaries: Sinusoids

Harmonically-related sines and cosines are *orthogonal*. Where k and m are positive integers:

$$\int_{t_0}^{t_0+T} \sin\left(\frac{2\pi k}{T}t\right)\cos\left(\frac{2\pi m}{T}t\right)\mathrm{d}t = 0$$

Preliminaries: Sinusoids

Where k and m are positive integers:

$$\int_{t_0}^{t_0+T} \cos\left(\frac{2\pi k}{T}t\right)\cos\left(\frac{2\pi m}{T}t\right)\mathrm{d}t = \begin{cases} T/2 & \text{if } k = m, \\ 0 & \text{otherwise} \end{cases}$$

Preliminaries: Sinusoids

Where k and m are positive integers:

$$\int_{t_0}^{t_0+T} \sin\left(\frac{2\pi k}{T}t\right)\sin\left(\frac{2\pi m}{T}t\right)\mathrm{d}t = \begin{cases} T/2 & \text{if } k = m, \\ 0 & \text{otherwise} \end{cases}$$

Finding Coefficients

Given these properties of sinusoids, and given a function:

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(\frac{2\pi kt}{T}\right) + \sum_{k=1}^{\infty} d_k \sin\left(\frac{2\pi kt}{T}\right)$$

how can we find the coefficients c_k and d_k?

c_0: The "DC" Term

For largely historical reasons, the constant c_0 is referred to as the "DC" term: it represents the constant offset of a signal (if any). Stated another way, it represents the *average value* of a signal over one period:

$$c_0 = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\mathrm{d}t$$

Computing the Other c_k Terms

We are assuming that our signal is comprised of harmonically-related sines and cosines. Here is an example signal:

$$f(t) = c_1 \cos(\omega_0 t) + d_1 \sin(\omega_0 t) + c_3 \cos(3\omega_0 t) + d_5 \sin(5\omega_0 t)$$

Let's imagine we wanted to determine the coefficient c_3, associated with $\cos(3\omega_0 t)$.

Our strategy will be to multiply $f(t)$ by $\cos(3\omega_0 t)$ and integrate the result over one period.

Check Yourself: How does this help us find c_3?

Computing the Other Terms

In general, we can compute the coefficients:

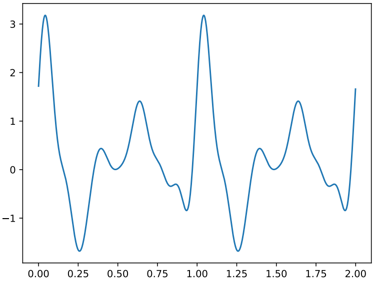
$$c_k = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos\left(\frac{2\pi kt}{T}\right)\mathrm{d}t = \frac{2}{T}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} f(t)\cos(k\omega_0 t)\mathrm{d}t$$

We can use similar logic to find the coefficients associated with the sine waves:

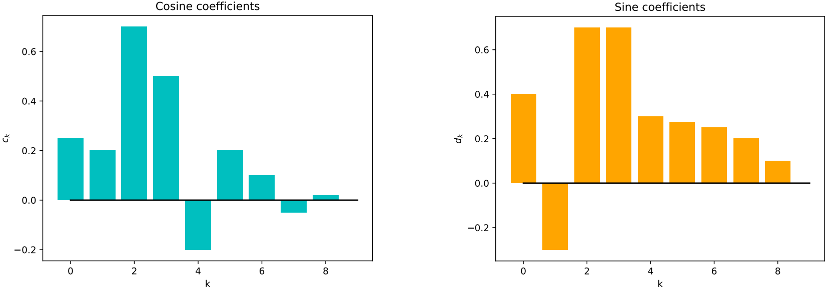
$$d_k = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin\left(\frac{2\pi kt}{T}\right)\mathrm{d}t = \frac{2}{T}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} f(t)\sin(k\omega_0 t)\mathrm{d}t$$

Two Different Views

Time domain:



Frequency domain:



Continuous-Time Fourier Series (Trig Form)

Synthesis Equation (making a signal from components):

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(\frac{2\pi k t}{T}\right) + \sum_{k=1}^{\infty} d_k \sin\left(\frac{2\pi k t}{T}\right)$$

Analysis Equations (finding the components):

$$c_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\,\mathrm{d}t$$

$$c_k = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos\left(\frac{2\pi k t}{T}\right) \mathrm{d}t \;; \qquad k > 0$$

$$d_k = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin\left(\frac{2\pi k t}{T}\right) \mathrm{d}t$$

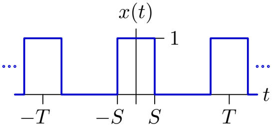
Warm Up

Find the Fourier series components (c_k and d_k) for

$$x(t) = x(t + 2\pi) = \cos(t)$$

Pulse Train

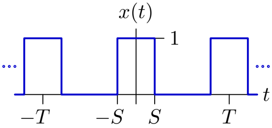
Find the Fourier series coefficients (c_k and d_k) for



Find the Fourier series coefficients and plot them versus k.

Pulse Train

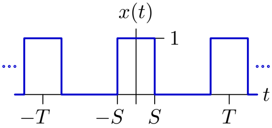
Find the Fourier series coefficients (c_k and d_k) for



What is the effect of scaling time (i.e., scaling S, T proportionally)?

Pulse Train

Find the Fourier series coefficients (c_k and d_k) for



What is effect of increasing only T? ... or decreasing only S?

Pulse Train

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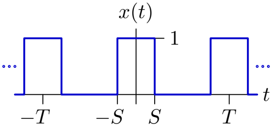
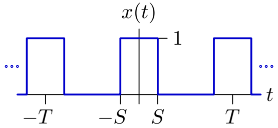


Synthesize with just N terms. What happens as you change N?

Pulse Train

Find the Fourier series coefficients (c_k and d_k) for

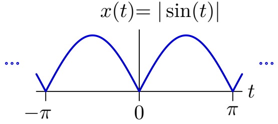


What would happen if you delayed $x(t)$ by $T/2$?

Rectified Sine Wave

Find the Fourier series coefficients (c_k and d_k) for

$$x(t) = x(t+\pi) = |\sin(t)|$$



Find the Fourier series coefficients and plot them versus k.

Rectified Sine Wave

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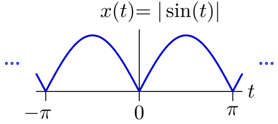

Synthesize with just N terms. Do you see Gibb's phenomenon?

Rectified Sine Wave

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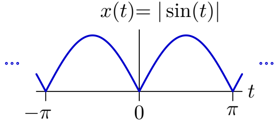
$$x(t) = x(t + \pi) = |\sin(t)|$$



Find the Fourier series coefficients for derivative of the rectified sine.

Will there be Gibb's?

Derivative

Consider:

$$x(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(\frac{2\pi k}{T} t\right) + \sum_{k=1}^{\infty} d_k \sin\left(\frac{2\pi k}{T} t\right)$$

What are the FSC associated with $x'(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}$?