6.003: Signal Processing

Sinusoidal Signals

Welcome to 6.003!

We will start @ 8:05am Eastern time

16 February 2021

Signals

6.003 is about **signal processing**.

Abstractly, a *signal* is a function that conveys information.

Examples:

- medical (EKG, EEG, MRI, ...)
- speech signals
- music
- seismic signals
- images
- video

Signal Processing

Signal processing is about extracting meaningful information from signals, and/or manipulating information in signals to produce new signals.

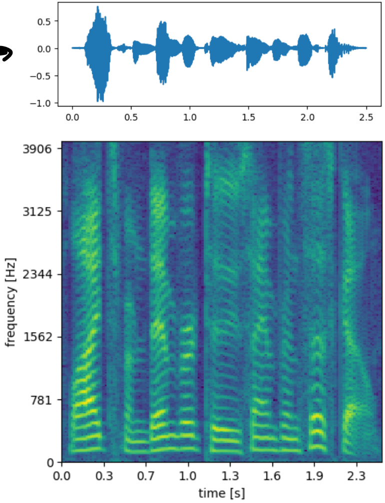
Transforms

Signals are functions that convey information. Sometimes, though, the straightforward way of representing a signal may not expose important properties of the signal.

It is often useful to have multiple different ways of looking at a signal.

Transforms

For example, consider the following two representations of a speech signal:



spectrogram

Transforms

We call such an alternative view of a signal a **transform**.

In 6.003, we will focus primarily on a family of transforms referred to as **Fourier** transforms, where signals are represented as sums of sinusoids, for example:

$$f(t) = \sum_{k=0}^{\infty} \left(c_k \cos k\omega_o t + d_k \sin k\omega_o t\right)$$

Sinusoids are interesting because they are prevalent in nature (and human perception), and because they have some nice mathemtical properties.

As such, much of our focus throughout 6.003 will be on sinusoidal functions (and, eventually, complex exponentials).

Preliminaries: Periodic Signals

A **periodic** signal is one that repeats after some amount of time T, such that, for all times t,

$$x(t) = x(t + T)$$

A function that is periodic in T is also periodic in $2T$, $3T$, $4T$, ... For a given signal, the *smallest* value T for which the above holds is called that signal's **fundamental period**.

The associated frequency, $\omega_0 = \frac{2\pi}{T}$, is the signal's **fundamental frequency**.

Frequencies are **harmonically related** if they are integer multiples of some fundamental frequency.

e.g., $\cos(\underline{2\omega_0} t)$ $\cos(\underline{7\omega_0} t)$

Preliminaries: CT Sinusoids

In general, a CT sinusoid has the form:

$$x(t) = A\cos(\underbrace{\omega t + \phi})$$

ω

- A is referred to as the *amplitude*
- ω is referred to as the *radian frequency*
- ϕ is referred to as a *phase offset*

units

A units of $x(\cdot)$

ω $\frac{\text{radians}}{\text{sec}}$

ϕ radians

t seconds

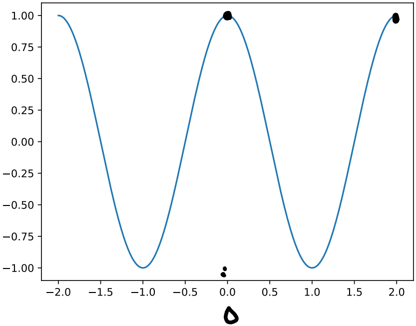
f $\frac{\text{cycles}}{\text{second}}$

2π $\frac{\text{radians}}{\text{cycle}}$

Check Yourself: Frequency

Consider $f(t) = \cos(\omega_1 t)$, shown below:

cos(ω_1t)



$\frac{2\pi \text{ rad}}{2 \text{ sec}}$

→ t (seconds)

Clicker: What is the value of ω_1?

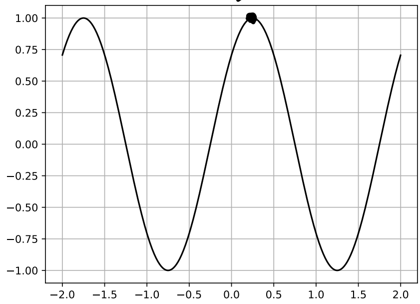
A. $1/2$ **B.** 1 **C.** 2 **D.** $\pi/2$ **E.** π **F.** 2π

rad/sec

In the chat: What are the units of ω_1?

Check Yourself: Phase

Consider a small, positive value a, where $0 \leq a \leq \pi$. Does the following represent $\cos(\omega_1 t + a)$ or $\cos(\omega_1 t - a)$?



$$\cos(0) = \cos(\underbrace{\omega_1 \times 0.25}_{\text{positive}} \pm \underbrace{a}_{\text{positive}})$$

Clicker:

A. $\cos(\omega_1 t + a)$

B. $\cos(\omega_1 t - a)$

Check Yourself: Sine Waves

In general, a CT sinusoid has the form:

$$x(t) = A\cos(\omega t + \phi) = A\cos\left(\frac{2\pi}{T}t + \phi\right)$$

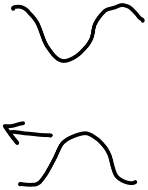
Can we represent $\sin\left(\frac{\pi}{3}t\right)$ in this form?

$$\sin(\omega t) = \cos\left(\omega t - \frac{\pi}{2}\right)$$

Clicker:

A. Yes

B. No



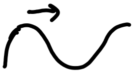
invert then shift left

$A = -1$

$\omega = \omega$

$\phi = +\frac{\pi}{2}$

or



no invert, shift right

$A = 1$

$\omega = \omega$

$\phi = -\frac{\pi}{2}$

If so, how? If not, why not?

Summing Sinusoids

Consider the following signal, consisting of the sum of two sinusoids:

$$f(t) = \underbrace{\cos\left(\frac{2\pi}{3}t + \frac{\pi}{4}\right)}_{T=3 \text{ seconds}} + \underbrace{\sin\left(\frac{\pi}{2}t\right)}_{T=4 \text{ seconds}}$$

$\frac{2\pi}{\omega}$

Is $f(\cdot)$ periodic?

0, 3, 6, 9, 12, 15, ...

0, 4, 8, 12, 16, ...

Clicker:

A. Yes

B. No

If so, Fund. period of $f(\cdot)$?

$\boxed{12 \text{ seconds}}$

$LCM(T_1, T_2)$

$$f(t) = \cos(\omega_1 t + \phi_1) + \cos(\omega_2 t + \phi_2)$$

is $f(t)$ guaranteed periodic?

No

~~cos~~

$T_1 = 2$ sec $\qquad T_2 = \sqrt{5}$ sec

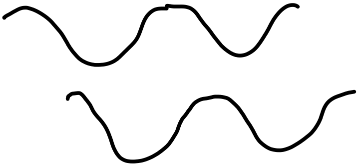
$f(\cdot)$ periodic iff $\dfrac{\omega_1}{\omega_2}$ is rational

Summing Sinusoids

What about the following signal?

$$f_2(t) = \cos(2\pi t) + 0.5\cos(2\pi t + \pi/4)$$

$T_1 = 1 \qquad T_2 = 1$



$$A_1 \cos(\omega t + \phi_1) + A_2 \cos(\omega t + \phi_2)$$

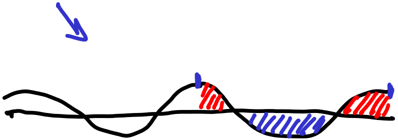
$$\Downarrow$$

$$B \cos(\omega t + \phi_3)$$

Integrating w/ Sinusoids

What is the result of evaluating the following definite integral?

$$\int_{t_0}^{t_0+T} \cos\left(\frac{2\pi}{T}t\right) \mathrm{d}t = ?$$



cancels out, $= 0$

Product of Harmonically-related sinusoids

What is the result of evaluating the following definite integral, where k and m are positive integers?

(Hint: it might help to look up some trig identities!)

$$\int_{t_0}^{t_0+T} \left(\sin\left(\frac{2\pi k}{T}t\right) \cos\left(\frac{2\pi m}{T}t\right) \right) \mathrm{d}t = ? \quad \boxed{0}$$

$$\sin(u)\cos(v) = \frac{1}{2}\left(\sin(u+v) + \sin(u-v)\right)$$

$$\underbrace{\frac{1}{2}\int_{t_0}^{t_0+T} \sin\left(\frac{2\pi(k+m)}{T}t\right)dt}_{0} + \underbrace{\frac{1}{2}\int_{t_0}^{t_0+T} \sin\left(\frac{2\pi(k-m)}{T}t\right)dt}_{0}$$

(integrating over integer # of periods)

Product of Harmonically-related sinusoids

What is the result of evaluating the following definite integral, where k and m are positive integers?

$$\int_{t_0}^{t_0+T} \left(\cos\left(\frac{2\pi k}{T}t\right) \cos\left(\frac{2\pi m}{T}t\right)\right) \mathrm{d}t = ?$$

$$\cos(u)\cos(v) = \frac{1}{2}\left(\cos(u-v) - \cos(u+v)\right)$$

$$\underline{\frac{1}{2}\int_{t_0}^{t_0+T} \cos\left(\frac{2\pi(k-m)}{T}t\right) dt} + \frac{1}{2}\int_{t_0}^{t_0+T} \cos\left(\frac{2\pi(k+m)}{T}t\right) dt$$

if $k=m$, $\frac{1}{2}\int_{t_0}^{t_0+T} dt = \frac{T}{2}$

$$= \begin{cases} 0, & \text{if } k \neq m \\ T/2, & \text{if } k = m \end{cases}$$

DT Sinusoids

In general, a DT sinusoid has the form:

$$x[n] = A\cos(\Omega n + \phi)$$

- A is referred to as the *amplitude*
- Ω is referred to as the *discrete frequency*
- ϕ is referred to as a *phase offset*

Importantly, n is always an integer!

(this has some really interesting consequences, which we'll explore in more detail in lecture 1B)