1 MRI

Recall that the measurements made by MRI machines are samples of the DFT $X[\cdot, \cdot]$ of some underlying image $x[\cdot, \cdot]$, which we recover via an inverse DFT. Recall also that, unlike many signals we have considered, this image’s spatial domain representation is generally complex-valued.

Throughout this problem, we will consider the same $256 \times 256$ example image from lecture, for which $|x[\cdot, \cdot]|$ is shown below:

As we discussed in lecture and recitation, an important ongoing area of research involves attempting to faithfully reconstruct an image using as few samples of $X[\cdot, \cdot]$ as possible. In this problem, we will consider several different attempts to reduce the scan time of the image above.

1.1 Part 1

Consider reducing the scanning time by only sampling half of the rows of $X[\cdot, \cdot]$, creating a new image whose DFT is given by:

$$X_2[k_r, k_c] = \begin{cases} 
X[k_r, k_c] & \text{if } k_r \text{ is odd} \\
0 & \text{otherwise}
\end{cases}$$

Which image on the facing page most closely matches $|x_2[\cdot, \cdot]|$, the magnitude of the spatial domain representation of this image?

Enter a single letter: \[ N \]

Also consider a different approach, where we still only sample half of the rows, but we fill in the missing rows via linear interpolation rather than leaving them as 0’s:

$$X_3[k_r, k_c] = \begin{cases} 
X[k_r, k_c] & \text{if } k_r \text{ is odd} \\
\frac{X[k_r + 1, k_c] + X[k_r - 1, k_c]}{2} & \text{otherwise}
\end{cases}$$

Which image on the facing page most closely matches $|x_3[\cdot, \cdot]|$, the magnitude of the spatial domain representation of this image?

Enter a single letter: \[ S \]
1.2 Part 2

Ben Bitdiddle suggests that a better way to cut down on scanning time would be to sample only half of the rows, but, particularly, to sample only where $0 \leq k_r \leq 128$, and then to use the conjugate symmetry of the DFT to fill in the missing values, i.e., for $-127 \leq k_r < 0$, set $X[k_r, k_c] = X^*[−k_r, −k_c]$.

Ben asserts that this approach will allow him to reconstruct $x[·, ·]$ exactly, while only explicitly sampling half of the DFT coefficients.

Is Ben’s assertion true? **Yes** or **No**: No

Briefly explain your reasoning:

The assumption that the DFT $X[·, ·]$ is conjugate symmetric is true if any only if the spatial-domain representation of the image $x[·, ·]$ is purely real.

So this approach would work if $x[·, ·]$ were real (regardless of the shape of $x[·, ·]$). However, because $x[·, ·]$ is actually complex valued, its DFT will not have this conjugate symmetry, and so Ben’s approach will not properly reconstruct the image.
Worksheet (intentionally blank)
Shapes

Each of the images below was created by computing the inverse DFT on a 24-by-24 array of DFT coefficients, of which at most 5 were non-zero. In each of these images, black represents a value of 0, and white represents a value of 1. The origin of each image is in its center, \( n_x \) increases to the right, and \( n_y \) increases downward.

For each image, enter the locations \((k_x, k_y)\) of all non-zero values in the associated DFT. If the image could not have been made from an array of the form described above, enter \textit{None} in the box instead.

Enter your answers on the facing page.
What are the locations of the nonzero values in the DFT associated with image A?
\[(0, 1), (0, -1), (0, 0)\]

What are the locations of the nonzero values in the DFT associated with image B?
\[(-2, -4), (2, 4), (0, 0)\]

What are the locations of the nonzero values in the DFT associated with image C?
\[(-1, 0), (1, 0), (0, 0), (0, -3), (0, 3)\]

What are the locations of the nonzero values in the DFT associated with image D?
\[(12, 0), (0, 0), (0, 12)\]

What are the locations of the nonzero values in the DFT associated with image E?
\[(-1, -1), (1, 1), (0, 0)\]

What are the locations of the nonzero values in the DFT associated with image F?
\[(-4, -2), (4, 2), (0, 0)\]

What are the locations of the nonzero values in the DFT associated with image G?
\[(0, 0)\]

What are the locations of the nonzero values in the DFT associated with image H?
\[(-2, 0), (2, 0), (0, -4), (0, 0), (0, 4)\]

What are the locations of the nonzero values in the DFT associated with image I?
\[(12, 0), (0, 0)\]

What are the locations of the nonzero values in the DFT associated with image J?
\[(-3, 0), (3, 0), (0, 0)\]

What are the locations of the nonzero values in the DFT associated with image K?
\[(0, 1), (0, -1), (0, 0), (0, 12)\]

What are the locations of the nonzero values in the DFT associated with image L?
\[(-3, 0), (3, 0), (0, 0), (0, -8), (0, 8)\]