Two-Dimensional DFT

\[ F[k_x, k_y] = \frac{1}{N_x N_y} \sum_{n_x=0}^{N_x-1} \sum_{n_y=0}^{N_y-1} f[n_x, n_y] e^{-j\left( \frac{2\pi k_x}{N_x} n_x + \frac{2\pi k_y}{N_y} n_y \right)} \]

\[ f[n_x, n_y] = \sum_{k_x=0}^{N_x-1} \sum_{k_y=0}^{N_y-1} F[k_x, k_y] e^{j\left( \frac{2\pi k_x}{N_x} n_x + \frac{2\pi k_y}{N_y} n_y \right)} \]
Simple Shapes

Find the 2D DFT of the following vertical bar.

Array indices in numpy are \([r, c]\), where \(r\) is row and \(c\) is column. The image is \(32 \times 32\) pixels. The bar is at \(c = 8\).
Simple Shapes

Find the 2D DFT of the following vertical bar.

\[
F[k_r, k_c] = \frac{1}{RC} \sum_{r=0}^{R-1} \sum_{c=0}^{C-1} f[r, c] e^{-j\left(\frac{2\pi k_r}{R} r + \frac{2\pi k_c}{C} c\right)}
\]

\[
= \frac{1}{32^2} \sum_{r=0}^{31} \sum_{c=0}^{31} \delta[c-8] e^{-j\left(\frac{2\pi k_r}{32} r + \frac{2\pi k_c}{32} c\right)}
\]

\[
= \frac{1}{32} \sum_{r=0}^{31} e^{-j\frac{2\pi k_r}{32} r} \frac{1}{32} \sum_{c=0}^{31} \delta[c-8] e^{-j\frac{2\pi k_c}{32} c} = \delta[k_r] \frac{1}{32} e^{-j\frac{2\pi k_c}{32} 8}
\]
Simple Shapes

Find the 2D DFT of the following vertical bar.

\[ f[r,c] = \delta[c-8] \]

\[ |F[k_r, k_c]| = \left| \delta[k_r] \frac{1}{32} e^{-j \frac{2\pi k_c 8}{32}} \right| \]

Frequency \([k_r, k_c]\) is often plotted with the origin in the center.

How does the \(e^{-j \frac{2\pi k_c 8}{32}}\) term contribute to the right panel?

Could you change \(f[r,c]\) so that \(F[k_r, k_c] = \frac{1}{32} \delta[k_r]\)? (no exponential)

Could you change \(f[r,c]\) so that the horizontal bar in \(F\) is at \(k_r = 8\)?
Simple Shapes

Find the 2D DFT of this image, where bars are at $c=0$ and $c=16$. 
Simple Shapes

Find the 2D DFT of this image, where bars are at $c=0$ and $c=16$.

$$\delta[c] \overset{\text{DFT}}{\leftrightarrow} \frac{1}{32} \delta[k_r]$$

$$\delta[c] + \delta[c-16] \overset{\text{DFT}}{\leftrightarrow} \frac{1}{32} \delta[k_r] + \frac{1}{32} e^{-j \frac{2\pi kc}{32} 16} \delta[k_r] = \frac{1}{32} \left(1 + (-1)^k\right) \delta[k_r]$$

$$= \begin{cases} 
\frac{1}{16} & \text{if } k_c \text{ is even and } k_r=0 \\
0 & \text{otherwise}
\end{cases}$$
Simple Shapes

Find the 2D DFT of this image, where bars are at \( c=0 \) and \( c=16 \).
Find the 2D DFT of the following image.
Simple Shapes

Find the 2D DFT of the following image.

\[
\begin{align*}
\delta[c] & \quad \xrightarrow{\text{DFT}} \quad \frac{1}{32} \delta[k_r] \\
\sum_{m=0}^{3} \delta[c-8m] & \quad \xrightarrow{\text{DFT}} \quad \frac{1}{32} \delta[k_r] \sum_{m=0}^{3} e^{-j \frac{2\pi k_c}{4} 8m} \\
& = \frac{1}{32} \delta[k_r] \sum_{m=0}^{3} e^{-j \frac{2\pi k_c}{4} m} = \frac{1}{8} \delta[k_r] \delta[k_c \mod 4]
\end{align*}
\]
Simple Shapes

Find the 2D DFT of the following image.

\[
\sum_{m=0}^{3} \delta[c-8m] \quad \xrightarrow{\text{DFT}} \quad \frac{1}{8} \delta[k_r] \delta[k_c \mod 4]
\]
Simple Shapes

Find the 2D DFT of the following image.
Find the 2D DFT of the following image.

\[
\begin{align*}
\delta[c] & \quad \longleftrightarrow \quad \frac{1}{32} \delta[k_r] \\
\sum_{m=0}^{7} \delta[c-4m] & \quad \longleftrightarrow \quad \frac{1}{32} \delta[k_r] \sum_{m=0}^{7} e^{-j \frac{2\pi k_c}{C} 4m} \\
& = \frac{1}{32} \delta[k_r] \sum_{m=0}^{7} e^{-j \frac{2\pi k_c}{8} m} = \frac{1}{4} \delta[k_r] \delta[k_c \mod 8]
\end{align*}
\]
Simple Shapes

Find the 2D DFT of the following image.

What’s the relation between the period in space (left) and the period in frequency (right)?