Convolution: Three Ways

The signal $x[n]$, defined below, is zero outside the indicated range.

Consider three ways to calculate the convolution of $x[n]$ with itself.

1. **direct convolution:**

   $$y_1[n] = (x * x)[n] = \sum_{m=-\infty}^{\infty} x[m]x[n-m]$$

2. **using DTFTs:**

   $$y_2[n] = \frac{1}{2\pi} \int_{2\pi} X^2(\Omega)e^{j\Omega n} d\Omega \quad \text{where} \quad X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

3. **using DFTs of length $N=16$:**

   $$y_3[n] = 16 \sum_{k=0}^{15} X^2[k]e^{j\frac{2\pi k}{16} n} \quad \text{where} \quad X[k] = \frac{1}{16} \sum_{n=0}^{15} x[n]e^{-j\frac{2\pi k}{16} n}$$
Convolution: Three Ways

The plots on the right show the **first ten samples** of five signals. Match the signals on the left with the corresponding plots on the right.

\[
y_1 = (x \ast x)
\]

\[
y_2 = \text{DTFT}^{-1}(X^2(\Omega))
\]

\[
y_3 = N \times \text{DFT}^{-1}(X^2[k])
\]
Convolution: Three Ways

Calculate \((x \ast x)[n]\) by direct convolution: flip and shift.

\[
y_1[n] = (x \ast x)[n] = \sum_{m=-\infty}^{\infty} x[m] x[n-m]
\]
Convolution: Three Ways

Calculate \((x*x)[n]\) by direct convolution: superposition.

\[ y_1[n] = (x*x)[n] = \sum_{m=-\infty}^{\infty} x[m]x[n-m] \]

\[ y_1[n] : \]

\[ x[0] \times x[n-0] : \]

\[ x[1] \times x[n-1] : \]

\[ x[8] \times x[n-8] : \]

Note: Superposition and flip-and-shift are equivalent methods. They always give the same answer.
Convolution: Three Ways

Plots on the left show the **first ten samples** of five signals. Match signals on the left with corresponding samples on the right.

\[ y_1 = (x \ast x) \]

\[ y_2 = \text{DTFT}^{-1}(X^2(\Omega)) \]

\[ y_3 = N \times \text{DFT}^{-1}(X^2[k]) \]
Convolution: Three Ways

Calculate \((x*x)[n]\) using DTFTs.

\[
X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = 1 + e^{-j\Omega} + e^{-j8\Omega}
\]

\[
X^2(\Omega) = \left(1 + e^{-j\Omega} + e^{-j8\Omega}\right)^2 = 1 + 2e^{-j\Omega} + e^{-j2\Omega} + 2e^{-j8\Omega} + 2e^{-j9\Omega} + e^{-j16\Omega}
\]

\[
y_2[n] = \frac{1}{2\pi} \int_{2\pi} X^2(\Omega)e^{j\Omega n} d\Omega
\]

\[
= \frac{1}{2\pi} \int_{2\pi} \left(1 + 2e^{-j\Omega} + e^{-j2\Omega} + 2e^{-j8\Omega} + 2e^{-j9\Omega} + e^{-j16\Omega}\right)e^{j\Omega n} d\Omega
\]

\[
= \delta[n] + 2\delta[n-1] + \delta[n-2] + 2\delta[n-8] + 2\delta[n-9] + \delta[n-16]
\]

Multiplying DTFTs is always equivalent to direct convolution.
Convolution: Three Ways

Plots on the left show the first ten samples of five signals. Match signals on the left with corresponding samples on the right.

\[
y_1 = (x \ast x)
\]

\[
y_2 = \text{DTFT}^{-1}(X^2(\Omega))
\]

\[
y_3 = N \times \text{DFT}^{-1}(X^2[k])
\]
Convolution: Three Ways

Calculate \((x \ast x)[n]\) using DFTs \((N = 16)\).

\[
X[k] = \frac{1}{16} \sum_{n=0}^{15} x[n] e^{-j \frac{2\pi k}{16} n} = \frac{1}{16} \left( 1 + e^{-j \frac{2\pi}{16}} + e^{-j \frac{2\pi 8}{16}} \right)
\]

\[
X^2[k] = \frac{1}{256} \left( 1 + 2e^{-j \frac{2\pi k}{16}} + e^{-j \frac{2\pi k}{16}} + 2e^{-j \frac{8\pi}{16}} + 2e^{-j \frac{9\pi}{16}} + e^{-j \frac{16\pi}{16}} \right)
\]

\[
y_3[n] = 16 \sum_{k=0}^{15} X^2[k] e^{j \frac{2\pi k}{16} n}
\]

\[
= \frac{1}{256} \sum_{k=0}^{15} \left( 2 + 2e^{-j \frac{2\pi k}{16}} + e^{-j \frac{2\pi k}{16}} + 2e^{-j \frac{8\pi}{16}} + 2e^{-j \frac{9\pi}{16}} \right) e^{j \frac{2\pi k}{16} n}
\]

\[
= 2\delta[n] + 2\delta[n-1] + \delta[n-2] + 2\delta[n-8] + 2\delta[n-9]
\]

Since \(N=16\), the sample at \(n=16\) in direct convolution aliases to \(n=0\).
Convolution: Three Ways

Plots on the left show the **first ten samples** of five signals. Match signals on the left with corresponding samples on the right.

\[ y_1 = (x \ast x) \]
\[ y_2 = \text{DTFT}^{-1}(X^2(\Omega)) \]
\[ y_3 = N \times \text{DFT}^{-1}(X^2[k]) \]
Circular Convolution

Multiplication of DFTs corresponds to **circular** convolution in time. Assume that $F[k]$ is the product of the DFTs of $f_a[n]$ and $f_b[n]$.

$$f[n] = \sum_{k=0}^{N-1} F[k]e^{j\frac{2\pi k}{N}n} = \sum_{k=0}^{N-1} F_a[k] F_b[k]e^{j\frac{2\pi k}{N}n}$$

$$= \sum_{k=0}^{N-1} F_a[k] \left( \frac{1}{N} \sum_{m=0}^{N-1} f_b[m]e^{-j\frac{2\pi k}{N}m} \right) e^{j\frac{2\pi k}{N}n}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] \sum_{k=0}^{N-1} F_a[k]e^{j\frac{2\pi k}{N}(n-m)}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] f_{ap}[n-m]$$

where $f_{ap}[n] = f_a[n \mod N]$ is a periodically extended version of $f_a[n]$. We refer to this as **circular** or **periodic** convolution:

$$\frac{1}{N} (f_a \otimes f_b)[n] \quad \overset{\text{DFT}}{\longleftrightarrow} \quad F_a[k] \times F_b[k]$$
Circular Convolution

Circular convolution is equivalent to conventional convolution followed by periodic summation of results back into base period.

\[
(f_a * f_b)[n] := (f_a \oplus f_b)[n] = (f_a \ast f_b)[n] \mod L
\]

\[
(f_a \oplus f_b)[n] := \sum_{m=0}^{L-1} f_a[n-m] \oplus f_b[m]
\]

\[
(f_a \oplus f_b)[n] := \sum_{m=0}^{n} f_a \mod L \oplus f_b[n-m] \mod L
\]
Circular Convolution

Circular convolution of two signals is equal to conventional convolution of one signal with a periodically extended version of the other.

\[
\begin{align*}
  f_a[n] & \quad \ast f_b[n] \\
0 & \quad 8 \\
\end{align*}
\]

\[
\begin{align*}
f_a[n] & \quad \ast f_b[n \mod N] \\
0 & \quad 8 \\
\end{align*}
\]

shifted 0:  
shifted 1:  
shifted 8:  

\[
(f_a \ast f_b)[n]:
-8 \quad 0 \quad 8 \quad 16 \quad 24
\]
Summary

One of the most useful properties of the DTFT is its filter property: convolution in time corresponds to multiplication in frequency.

\[(f \ast g)[n] \quad \text{DTFT} \quad F(\Omega)G(\Omega)\]

The DFT is slightly more complicated since the DFT is equivalent to the DTFS of a periodically extended version of \(x[n]\):

\[x[n] = x[n + mN] \quad \text{for all integers } m\]

A result of this periodicity is that the convolution that results when two DFTs are multiplied is also periodic.

We refer to this type of convolution as “circular convolution.”

\[(f \bigcirc g)[n] \quad \text{DFT} \quad F[k]G[k]\]