6.003: Signal Processing

Sinusoids and Fourier Series

February 6, 2020
Fourier Series (Trigonometric Form)

If \( f(t) \) is expressed as a Fourier series

\[
f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} \left( c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \right)
\]

the Fourier coefficients are given by

\[
c_0 = \frac{1}{T} \int_T f(t) \, dt
\]

\[
c_k = \frac{2}{T} \int_T f(t) \cos(k\omega_o t) \, dt; \quad k = 1, 2, 3, \ldots
\]

\[
d_k = \frac{2}{T} \int_T f(t) \sin(k\omega_o t) \, dt; \quad k = 1, 2, 3, \ldots
\]
Two Pulses

Let $f_1(t)$ represent the following function, which is periodic in $T = 7$:

$$f_1(t) = c_0 + \sum_{k=1}^{\infty} \left( c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t) \right)$$

Determine a Fourier series of the following form for $f_1(t)$. 

$$f_1(t) = c_0 + \sum_{k=1}^{\infty} \left( c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t) \right)$$
Two Pulses

The average value is $c_0 = \frac{1}{T} \int_T f_1(t) \, dt$.

There are many equivalent ways to integrate over a period:

\[
\int_T dt = \int_0^7 dt = \int_{-7/2}^{7/2} dt = \cdots
\]

All of these are easy. The result is $c_0 = \frac{2}{7}$. 
Two Pulses

For \( k \geq 1 \):

\[
c_k = \frac{2}{T} \int_T^T f_1(t) \cos(k\omega_0 t) \, dt
\]

Integrating over symmetric limits simplifies the math (slightly):

\[
c_k = \frac{2}{7} \int_{-2}^{2} \cos(k\omega_0 t) \, dt - \frac{2}{7} \int_{-1}^{1} \cos(k\omega_0 t) \, dt
\]

\[
= \frac{2}{7} \sin(k\omega_0 t) \bigg|_{-2}^{2} - \frac{2}{7} \sin(k\omega_0 t) \bigg|_{-1}^{1}
\]

\[
= \frac{2}{\pi k} \left( \sin \frac{4k\pi}{7} - \sin \frac{2k\pi}{7} \right)
\]

Demonstrate other ways to do this integration, e.g.,

\[
\int_{-2}^{1} dt + \int_{1}^{2} dt \quad \text{or} \quad \int_{1}^{2} dt + \int_{6}^{7} dt \quad \text{or} \quad 2 \int_{1}^{2} dt
\]

Since \( f_1(t) \) is symmetric, and harmonics of the sine function are antisymmetric, the \( d_k \) coefficients are all zero.

\[
d_k = \frac{2}{T} \int_T^T f_1(t) \sin(k\omega_0 t) \, dt = 0
\]
Checking with Python

Sum the first 100 elements of series:

```python
from math import sin, pi
from matplotlib.pyplot import plot, show
x = []
y = []
omega0 = 2*pi/7
t = -7
while t<7:
    x.append(t)
    y.append(2/7+sum([2*(sin(4*pi*k/7)-sin(2*pi*k/7))/pi/k*cos(k*omega0*t)
                     for k in range(1,100)]))
    t += 0.01
plot(x,y)
show()
```
Let $f_2(t)$ represent the following function, which is periodic in $T = 7$:

$$f_2(t) = \begin{cases} 1 & -7 \leq t < -2 \\ -1 & -2 \leq t < -1 \\ 1 & -1 \leq t < 1 \\ -1 & 1 \leq t < 2 \\ 1 & 2 \leq t < 7 \\ -1 & 7 \leq t < 7 \\ \vdots \\ \end{cases}$$

Find $\omega_o$ and the Fourier series coefficients $c_k$ and $d_k$ so that

$$f_2(t) = \sum_{k=0}^{\infty} \left( c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t) \right)$$
Opposite Pulses

The previous problem was symmetric about $t = 0$ so there were only cosine terms.

This problem is antisymmetric about $t = 0$ so there are only sine terms.

Do the calculation using several different regions of integration.
Single Pulse

Let $f_3(t)$ represent the following function, which is periodic in $T = 7$:

Determine the Fourier series coefficients for $f_3(t)$.
Discuss the relation(s) among the Fourier series coefficients of $f_1(t)$, $f_2(t)$, and $f_3(t)$.
Discuss the relation(s) among $f_1(t)$, $f_2(t)$, and $f_3(t)$.
Check With Python

It’s interesting that overshoots move around. They are "glued" to the discontinuities.
### Trig Table

\[
\begin{align*}
\sin(a+b) &= \sin(a) \cos(b) + \cos(a) \sin(b) \\
\sin(a-b) &= \sin(a) \cos(b) - \cos(a) \sin(b) \\
\cos(a+b) &= \cos(a) \cos(b) - \sin(a) \sin(b) \\
\cos(a-b) &= \cos(a) \cos(b) + \sin(a) \sin(b) \\
\tan(a+b) &= \frac{\tan(a)+\tan(b)}{1-\tan(a) \tan(b)} \\
\tan(a-b) &= \frac{\tan(a)-\tan(b)}{1+\tan(a) \tan(b)} \\
\sin(A) + \sin(B) &= 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) \\
\sin(A) - \sin(B) &= 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right) \\
\cos(A) + \cos(B) &= 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) \\
\cos(A) - \cos(B) &= -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right) \\
\sin(a+b) + \sin(a-b) &= 2 \sin(a) \cos(b) \\
\sin(a+b) - \sin(a-b) &= 2 \cos(a) \sin(b) \\
\cos(a+b) + \cos(a-b) &= 2 \cos(a) \cos(b) \\
\cos(a+b) - \cos(a-b) &= -2 \sin(a) \sin(b) \\
2 \cos(A) \cos(B) &= \cos(A-B) + \cos(A+B) \\
2 \sin(A) \sin(B) &= \cos(A-B) - \cos(A+B) \\
2 \sin(A) \cos(B) &= \sin(A+B) + \sin(A-B) \\
2 \cos(A) \sin(B) &= \sin(A+B) - \sin(A-B)
\end{align*}
\]