Practice Problems for Quiz #2

The questions were taken from multiple previous exams.
1 DTFT

Part 1.

Consider a signal \( x_1 \) given by the following expression:

\[
x_1[n] = \begin{cases} 
1 & \text{if } |n| \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]

Sketch the magnitude and angle of \( X_1 \) on the axes below. Label key points.

Starting with the definition of the Fourier Transform,

\[
X_1(\Omega) = \sum_{n=-\infty}^{\infty} x_1[n]e^{-j\Omega n} = \sum_{n=-1}^{1} e^{-j\Omega n} = e^{j\Omega} + 1 + e^{-j\Omega} = 1 + 2\cos\Omega
\]

This function is plotted below.

Since \( X_1(\Omega) \) is real-valued, the magnitude of \( X_1(\Omega) \) is equal to the absolute value of \( X_1(\Omega) \) as shown in the top left plot.

Because \( X_1(\Omega) < 0 \) for \( \frac{2\pi}{3} < |\Omega| + 2\pi i < \frac{4\pi}{3} \) (where \( i \) is an integer) the angle of \( X_1(\Omega) \) is \( \pi \), as shown in the top right plot.
Part 2.
Consider a signal $x_2$ described by the following diagram:

Determine a closed-form expression for the Fourier transform of $x_2$.

$$X_2(\Omega) = \frac{6 + 2e^{-j\Omega} - 2e^{-j2\Omega} - e^{-j3\Omega}}{(3 - e^{-j2\Omega})(2 - e^{-j2\Omega})}$$
2 More Than Meets The Eye

Part 1. Consider the function $x_1$ described by the following expression and plot:

$$ x_1(t) = 3te^{-2|t|} $$

Determine the Fourier transform $X_1$ of this signal. Express your answer in closed form.

$$ X_1(\omega) = \frac{-24j\omega}{(4 + \omega^2)^2} $$

\[
\begin{align*}
y_1(t) &= e^{-t}u(t) \quad \xrightarrow{\text{FT}} \quad Y_1(\omega) = \int_0^\infty e^{-t}e^{-j\omega t}dt = \frac{1}{1+j\omega} \\
y_2(t) &= e^{-2t}u(t) = y_1(2t) \quad \xrightarrow{\text{FT}} \quad Y_2(\omega) = \frac{1}{2}Y_1\left(\frac{\omega}{2}\right) = \frac{1}{2+j\omega} \\
y_3(t) &= te^{-2t}u(t) = ty_2(t) \quad \xrightarrow{\text{FT}} \quad Y_3(\omega) = j\frac{d}{d\omega}Y_2(\omega) = \frac{1}{(2+j\omega)^2} \\
y_4(t) &= te^{-2|t|} = te^{-2t}u(t) + te^{2t}u(-t) = y_3(t) - y_3(-t) \quad \xrightarrow{\text{FT}} \quad Y_4(\omega) = \frac{1}{(2+j\omega)^2} - \frac{1}{(2-j\omega)^2} \\
x_1(t) &= 3te^{-2|t|} = 3y_4(t) \quad \xrightarrow{\text{FT}} \quad X_1(\omega) = \frac{3}{(2+j\omega)^2} - \frac{3}{(2-j\omega)^2} \\
\end{align*}
\]
**Part 2.** Now consider a signal $x_2$ whose Fourier transform $X_2$ is given by the following expression:

$$X_2(\omega) = 3\omega e^{-2|\omega|}$$

Determine a closed-form expression for $x_2(t)$.

$$x_2(t) = \frac{j12t}{\pi(4 + t^2)^2}$$

**Duality:** If $x(t) \xleftrightarrow{FT} X(\omega)$ then

$$X(t) \xleftrightarrow{FT} 2\pi x(-\omega)$$

From part 1,

$$3te^{-2|t|} \xleftrightarrow{FT} -\frac{24j\omega}{(4 + \omega^2)^2}$$

By duality,

$$-\frac{24jt}{(4 + t^2)^2} \xleftrightarrow{FT} 2\pi(-3\omega e^{-2|\omega|})$$

$$\frac{j12t}{\pi(4 + t^2)^2} \xleftrightarrow{FT} 3\omega e^{-2|\omega|}$$
Part 3
Assume that a function $x_3$ has a Fourier transform given by $X_3$.
Let $y_3$ be defined in terms of $x_3$, as follows:
$$y_3(t) = \dot{x}_3(3(t + 5))$$
where $\dot{x}_3(t)$ is the time derivative of $x_3(t)$.

Find $Y_3(\omega)$ in terms of $X_3$:
$$Y_3(\omega) = \frac{1}{3} j \omega e^{j5\omega} X_3(\frac{\omega}{3})$$

Let
$$z(t) = \dot{x}(t) = \frac{dx(t)}{dt}.$$
Then
$$Z(\omega) = j \omega X(\omega).$$
We can express $y(t)$ in terms of $z$ as
$$y(t) = z(3(t + 5))$$
and then
$$Y(\omega) = \int y(t) e^{-j\omega t} dt$$
$$= \int z(3(t + 5)) e^{-j\omega t} dt$$
Let $\tau = 3(t + 5)$ then $d\tau = 3 dt$ and
$$Y(\omega) = \int z(\tau) e^{-j\omega(\frac{\tau}{3} - 5)} \frac{1}{3} d\tau$$
$$= \frac{1}{3} e^{j5\omega} \int z(\tau) e^{-j\frac{\omega}{3} \tau} d\tau$$
$$= \frac{1}{3} e^{j5\omega} Z(\frac{\omega}{3})$$
$$= \frac{1}{3} e^{j5\omega} \frac{\omega}{3} X(\frac{\omega}{3})$$
3 Fourier Transform

Part 1
The continuous-time signal \( x_1(t) \) is defined by the following plot

\[ x_1(t) \]

and is zero outside the indicated range of \( t \).

Compute \( X_1(\omega) \), the Fourier Transform of \( x_1(t) \).

\[
X_1(\omega) = \frac{2 \sin 3\omega}{\omega} - \frac{2 \sin 2\omega}{\omega}
\]

The signal \( x(t) \) can be written as the difference between a rectangular pulse that extends from \(-3\) to \(3\) and a rectangular pulse that extends from \(-2\) to \(2\). The Fourier transform of the former is

\[
\int_{-3}^{3} e^{-j\omega t} \, dt = \frac{e^{-j\omega t}}{-j\omega} \bigg|_{-3}^{3} = 2 \frac{\sin 3\omega}{\omega}
\]

Similarly, the Fourier transform of the latter is \( 2 \frac{\sin 2\omega}{\omega} \). Thus the total answer is

\[
X(\omega) = 2 \frac{\sin 3\omega}{\omega} - 2 \frac{\sin 2\omega}{\omega}
\]
Part 2

Determine the Fourier transform of \( x_2(t) \) given by the following expression

\[
x_2(t) = \begin{cases} 
e^{|t|} & \text{if } -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}
\]

and illustrated below.

Enter a closed-form expression for the Fourier transform in the box below, both in complex exponential form and in terms of sines and cosines.

\[
X_2(\omega) = \frac{2e \cos \omega + 2e \omega \sin \omega - 2}{1 + \omega^2}
\]

Calculate the Fourier transform of the right half of this function:

\[
X_r(\omega) = \int_0^1 e^{t} e^{-j\omega t} dt = \int_0^1 e^{(1-j\omega)t} dt = \left. \frac{e^{(1-j\omega)t}}{1-j\omega} \right|_0^1 = \frac{e^{(1-j\omega)} - 1}{1 - j\omega}
\]

The Fourier transform of the left part is

\[
X_l(\omega) = X_r(-\omega) = \frac{e^{(1+j\omega)} - 1}{1 + j\omega}
\]

Therefore

\[
X(\omega) = X_r(\omega) + X_l(\omega) = \frac{e^{(1-j\omega)} - 1}{1 - j\omega} + \frac{e^{(1+j\omega)} - 1}{1 + j\omega} = \frac{2e \cos \omega + 2e \omega \sin \omega - 2}{1 + \omega^2}
\]
Part 3

Consider a discrete-time function $x_3[n]$ given by the following expression:

$$
x_3[n] = \begin{cases} 
-1 & \text{if } n = 0 \\
1 & \text{if } n = 1 \\
0 & \text{otherwise}
\end{cases}
$$

Sketch the magnitude and angle of the the Fourier transform of $x_3$, $X_3(\Omega)$, on the axes below. Label key points.

Straightforward application of the DTFT analysis equation yields:

$$X_3(\Omega) = \sum_{n=-\infty}^{\infty} x_3[n]e^{-j\Omega n} = e^{-j\Omega} - 1$$

To get a better sense for the shape of these curves, we can pull out a common factor to put this closer to a form we have seen before:

$$X_3(\Omega) = e^{-j\Omega/2} \left( 2j \sin(\Omega/2) \right)$$

From here, it is easier to see that the phase is changing linearly with $\Omega$ (with the exception of the discontinuities at $\Omega = 0, 2\pi, 4\pi, \ldots$ where the sine wave changes signs), and that the magnitude is $2|\sin(\Omega/2)|$.

You may also find it helpful to first plot the magnitude and phase of $-2j \sin(\Omega/2)$, and then to think about how the $e^{-j\Omega/2}$ term affects those plots.
4 Frequency Response

Part 1. Let $h_1[n]$ represent the unit sample response of a linear, time-invariant system

$$x[n] \xrightarrow{\text{LTI system}} y[n] = (x * h_1)[n]$$

where

$$h_1[n] = \begin{cases} 1 & \text{if } n = 0 \text{ or } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine the magnitude and angle of the frequency response of this filter, and enter expressions for them in the boxes below.

magnitude = $2|\cos \frac{\Omega}{2}|$

angle = $-\frac{\Omega}{2}$

$$H_1(\Omega) = 1 + e^{-j\Omega} = 2e^{-j\Omega/2}(e^{j\Omega/2} + e^{-j\Omega/2}) = 2e^{-j\Omega/2}\cos \frac{\Omega}{2}$$

Part 2. Let $h_2[n]$ represent an alternative unit sample response where

$$h_2[n] = \begin{cases} -j & \text{if } n = 0 \\ j & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the frequency response of this filter and plot its magnitude and phase on the axes below. Label all important frequencies, magnitudes, and angles.
\[ H_2(\Omega) = -j + je^{-j\Omega} = 2e^{-j\Omega/2} \left( \frac{e^{j\Omega/2} - e^{-j\Omega/2}}{2j} \right) = 2e^{-j\Omega/2} \sin \frac{\Omega}{2} \]

\[ |H_2(\Omega)| = \left| \sin \frac{\Omega}{2} \right| \]

\[ \angle H_2(\Omega) = \begin{cases} -\Omega/2 & \text{if } 0 < \Omega < 2\pi \\ -\Omega/2 - \pi & \text{if } -2\pi < \Omega < 0 \end{cases} \]
5 Phase Matching

The following periodic signal $x(t)$ has period $T = 8$.

$$x(t) = x(t - 8)$$

The magnitude and phase of the Fourier series coefficients $a_k$ of $x(t)$ are given below.

Determine which angle function from the next page corresponds to each of these signals:
Note: $\times$ means the angle is undefined because the magnitude of that coefficient is zero.
6 Frequency Response

Let \( X(\omega) \) represent the following Fourier Transform (CTFT) of the continuous-time signal \( x(t) \).

\[
X(\omega) = \begin{cases} 
e^{j\omega} & \text{if } -\pi \leq \omega \leq \pi \\ 0 & \text{otherwise} \end{cases}
\]

**Part 1.** Determine a closed-form expression for \( x(t) \).

\[
x(t) = \frac{\sin(\pi(t+1))}{\pi(t+1)}
\]

Plot \( x(t) \) on the axes below. Indicate the times and values of all key features of your plot.
Part 2. Let $y[n]$ represent the discrete-time signal that results from sampling $x(t)$ once every $T = \frac{1}{2}$ seconds. Determine $Y(\Omega)$, which represents the discrete-time Fourier transform (DTFT) of $y[n]$. Plot the magnitude and angle of $Y(\Omega)$ on the axes below.

\[
\begin{align*}
|Y(\Omega)| & \quad \Omega \\
-2\pi & \quad -\pi & \quad \pi/2 & \quad \pi & \quad 3\pi/2 & \quad 2\pi
\end{align*}
\]

\[
\begin{align*}
\angle Y(\Omega) & \quad \Omega \\
-2\pi & \quad -3\pi/2 & \quad -\pi & \quad -\pi/2 & \quad \pi/2 & \quad \pi & \quad 3\pi/2 & \quad 2\pi
\end{align*}
\]


\[
y[n] = \sin \frac{\pi(n+2)}{2}\frac{\pi(n+2)}{2}
\]

\[
y[n] = x(nT) = \left. \frac{\sin(\pi(t+1))}{\pi(t+1)} \right|_{t=nT} = \frac{\sin(\pi(\frac{n}{2} + 1))}{\pi(\frac{n}{2} + 1)} = 2\sin\left(\frac{\pi}{2}(n+2)\right)
\]

Since $y[n]$ is a sinc function of $n$, $Y(\Omega)$ is a lowpass filter with cutoff frequency $\frac{\pi}{2}$ and DC value 2. There is also a time shift of 2 since $n$ has been replaced by $n + 2$. This time shift introduces a phase lead of $2\Omega$. The final answer is

\[
Y(\Omega) = \begin{cases} 
2e^{j2\Omega} & \text{if } -\pi < \Omega + 2\pi m < \pi \text{ for some integer } m \\
0 & \text{otherwise}
\end{cases}
\]
Part 4. Let $z[n]$ represent the discrete-time signal that results from sampling $x(t)$ once every $T = 1$ second. Determine $Z(\Omega)$, which represents the discrete-time Fourier transform (DTFT) of $z[n]$. Plot the magnitude and angle of $Z(\Omega)$ on the axes below.

\[|Z(\Omega)|\]

\[-2\pi \quad -\frac{3\pi}{2} \quad -\pi \quad -\frac{\pi}{2} \quad \frac{\pi}{2} \quad \pi \quad \frac{3\pi}{2} \quad 2\pi\]

\[\angle Z(\Omega)\]

\[-2\pi \quad -\frac{3\pi}{2} \quad -\pi \quad -\frac{\pi}{2} \quad \frac{\pi}{2} \quad \pi \quad \frac{3\pi}{2} \quad 2\pi\]

Part 5. Determine an expression for $z[n]$.

\[z[n] = \begin{bmatrix} \frac{\sin(\pi(n+1))}{\pi(n+1)} = \delta[n + 1] \end{bmatrix}\]

\[z[n] = x(nT) = \frac{\sin(\pi(t + 1))}{\pi(t + 1)} \bigg|_{t=nT} = \frac{\sin(\pi(n+1))}{\pi(n+1)} = \delta[n + 1]\]

The DTFT of $\delta[n]$ is 1. The time shift introduces a phase lead of $\Omega$. Thus

\[Z(\Omega) = e^{j\Omega}\]
7 Chirp

Part 1.

Consider an LTI system whose unit-sample response is given by:

\[ h[n] = \begin{cases} 
\cos \left( \frac{\pi}{10} n \right) & \text{if } 0 \leq n \leq n_{\text{off}} \\
0 & \text{otherwise}
\end{cases} \]

where \( n_{\text{off}} \) is chosen so that exactly four full periods of the cosine can be seen in the unit sample response.

For some input \( x[n] \), this system’s output will consist of six full periods of a cosine at the same frequency, followed by zeros:

Determine this signal \( x[n] \). In the boxes below, enter both the value of \( n \) and the associated value of \( x[n] \) for the first 10 nonzero values of \( x[n] \). The boxes below should contain only numbers. If there are fewer than 10 nonzero values in \( x[n] \), write None in the remaining boxes.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x[n] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>80</td>
<td>1</td>
</tr>
<tr>
<td>120</td>
<td>-1</td>
</tr>
<tr>
<td>160</td>
<td>1</td>
</tr>
<tr>
<td>200</td>
<td>-1</td>
</tr>
<tr>
<td>240</td>
<td>1</td>
</tr>
<tr>
<td>280</td>
<td>-1</td>
</tr>
<tr>
<td>320</td>
<td>1</td>
</tr>
<tr>
<td>360</td>
<td>-1</td>
</tr>
<tr>
<td>400</td>
<td>1</td>
</tr>
</tbody>
</table>

Explanation on following page.
This problem is similar to the “echo cancellation” lab, and so we can take a similar approach here to how we approached that problem. We can start by thinking about $x[n] = \delta[n]$. In this case, $y[n] = h[n]$, i.e., a cosine that goes through four full periods.

Adding another delta at $n = 80$ would cause the cosine to continue for another four full periods. That is, if $x[n] = \delta[n] + \delta[n - 80]$, we would have an output $y[n]$ that looked like eight full periods of the cosine.

Since we would like only six periods of the cosine, we need something to cancel out the samples between $n = 120$ and $n = 160$. We can do this by adding an upside-down delta at $n = 120$. At this point, $x[n] = \delta[n] + \delta[n - 80] - \delta[n - 120]$. The response to this signal will be six full periods of the cosine, followed by 40 zeros, followed by another 2 periods of an upside-down cosine.

We can add another delta at $n = 160$ to cancel out those 2 periods of upside-down cosine, but that leaves another 2 periods of the cosine starting at $n = 200$. Another upside-down delta at that point cancels those, but leaves another 2 periods starting at $n = 240$. Ultimately, we need an infinite sequence of deltas (with alternating signs) to make sure $y[n]$ is zero for all $n \geq 120$. 

Part 2.

We could compute this response by multiplying the DTFTs of these two signals, $X(\Omega)$ and $H(\Omega)$. Find closed-form expressions for $X(\Omega)$ and $H(\Omega)$ and enter them in the boxes below.

Hint: \[ \sum_{n=0}^{N} a^n = \frac{1 - a^{N+1}}{1 - a} \]

\[
H(\Omega) = \frac{1}{2} \left( \frac{1 - e^{j(\frac{\pi}{10} - \Omega)80}}{1 - e^{j(\frac{\pi}{10} - \Omega)}} + \frac{1 - e^{-j(\frac{\pi}{10} + \Omega)80}}{1 - e^{-j(\frac{\pi}{10} + \Omega)}} \right)
\]

\[
H(\Omega) = \sum_{n=0}^{79} \cos \left( \frac{\pi}{10} n \right) e^{-j\Omega n} = \sum_{n=0}^{79} \frac{1}{2} \left( e^{j\frac{\pi}{10} n} + e^{-j\frac{\pi}{10} n} \right) e^{-j\Omega n}
\]

\[
= \sum_{n=0}^{79} \frac{1}{2} \left( e^{j\left(\frac{\pi}{10} - \Omega\right) n} + e^{-j\left(\frac{\pi}{10} + \Omega\right) n} \right)
\]

\[
= \frac{1}{2} \left( \sum_{n=0}^{79} \left( e^{j\left(\frac{\pi}{10} - \Omega\right) n} \right) + \sum_{n=0}^{79} \left( e^{-j\left(\frac{\pi}{10} + \Omega\right) n} \right) \right)
\]

Simplifying using the hint from above, we find:

\[
H(\Omega) = \frac{1}{2} \left( \frac{1 - e^{j(\frac{\pi}{10} - \Omega)80}}{1 - e^{j(\frac{\pi}{10} - \Omega)}} + \frac{1 - e^{-j(\frac{\pi}{10} + \Omega)80}}{1 - e^{-j(\frac{\pi}{10} + \Omega)}} \right)
\]
\[ X(\Omega) = 1 + \frac{e^{-j80\Omega}}{1 + e^{-j40\Omega}} \]

From the preceding page, we have:

\[ x[n] = \delta[n] + \sum_{m=0}^{\infty} (-a)^m \delta[n - 80 - 40m] = \delta[n] + \delta[80] - \delta[120] + \delta[160] - \delta[200] + \ldots \]

We can start by applying the DTFT equation directly to the form on the right, and then do some manipulation and close the infinite sum:

\[
X(\Omega) = 1 + e^{-j80\Omega} - e^{-j120\Omega} + e^{-j160\Omega} - e^{-j200\Omega} + \ldots \\
= 1 + e^{-j80\Omega} (1 - e^{-j40\Omega} + e^{-j80\Omega} - e^{-j120\Omega} + \ldots) \\
= 1 + e^{-j80\Omega} \left( \sum_{m=0}^{\infty} (-1)^m e^{-j40\Omega m} \right) \\
= 1 + e^{-j80\Omega} \left( \frac{1}{1 + e^{-j40\Omega}} \right)
\]
8 Steps

Part 1. Let $x[n]$ represent the following discrete-time signal

$$x[n] = \begin{cases} 
0 & \text{for } n < 0 \\
a^0 & \text{for } n = 0, 1, 2 \\
a^1 & \text{for } n = 3, 4, 5 \\
a^2 & \text{for } n = 6, 7, 8 \\
\ldots 
\end{cases}$$

where $a$ is a real number between 0 and 1, as shown in the plot below.

Determine a closed form expression for $X(\Omega)$, which is the discrete-time Fourier transform of $x[n]$.

$$X(\Omega) = \frac{1 + e^{-j\Omega} + e^{-j2\Omega}}{1 - ae^{-3\Omega}}$$

Solution on following page.
Consider a related, but simpler signal:

\[ x'[n] = \begin{cases} 
    a^{(n/3)} & \text{if } n \mod 3 \equiv 0 \\
    0 & \text{otherwise}
\end{cases} \]

The Fourier transform of this signal is:

\[ X'(\Omega) = \sum_{m=0}^{\infty} a^m e^{-j3\Omega m} = \frac{1}{1 - ae^{-j3\Omega}} \]

\[ x[n] = x'[n] + x'[n - 1] + x'[n - 2], \text{ so, by linearity and the time shift property:} \]

\[ X(\Omega) = \left(1 + e^{-j\Omega} + e^{-j2\Omega}\right) X'(\Omega) = \frac{1 + e^{-j\Omega} + e^{-j2\Omega}}{1 - ae^{-j3\Omega}} \]

Alternatively, apply the formula directly and simplify:

\[ X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \]

\[ = \sum_{m=0}^{\infty} a^m \left( e^{-j\Omega 3m} + e^{j\Omega (3m+1)} + e^{j\Omega (3m+2)} \right) \]

\[ = \sum_{m=0}^{\infty} a^m e^{-j\Omega 3m} \left( 1 + e^{j\Omega} + e^{j2\Omega} \right) \]

\[ = \frac{1 + e^{-j\Omega} + e^{-j2\Omega}}{1 - ae^{-j3\Omega}} \]
**Part 2.** Let $x(t)$ represent the following continuous-time signal

$$x(t) = \begin{cases} 
0 & \text{for } t < 0 \\
a^0 & \text{for } 0 \leq t < 3 \\
a^1 & \text{for } 3 \leq t < 6 \\
a^2 & \text{for } 6 \leq t < 9 \\
& \cdots 
\end{cases}$$

where $a$ is a real number between 0 and 1, as shown in the plot below.

Determine a closed-form expression for $X(\omega)$, which is the continuous-time Fourier transform of $x(t)$.

$$X(\omega) = \left( \frac{1}{j\omega} \right) \left( \frac{1 - e^{-j3\omega}}{1 - ae^{-j3\omega}} \right)$$

Solution on following page.
Consider a simpler signal: 
\( x'(t) = \begin{cases} 
1 & \text{if } 0 < n < 3 \\
0 & \text{otherwise} 
\end{cases} \)

Then we have 
\( x(t) = x'(t) + ax'(t-3) + a^2x'(t-6) + a^3x'(t-9) + \ldots \), so, by linearity and the time shift property:

\[
X(\omega) = X'(\omega) + ae^{-j3\omega}X'(\omega) + a^2e^{-j6\omega}X'(\omega) + \ldots \\
= X'(\omega)(1 + ae^{-j3\omega} + a^2e^{-j6\omega} + \ldots) \\
= X'(\omega) \left( \frac{1}{1 - ae^{-j3\omega}} \right)
\]

Solving for \( X'(\omega) \), we find:

\[
X'(\omega) = \int_0^3 a^0 e^{-j\omega t} dt = \frac{e^{-j\omega t}}{-j\omega} \bigg|_0^3 = \frac{1 - e^{-j3\omega}}{j\omega}
\]

So we have 
\[
X(\omega) = X'(\omega) \left( \frac{1}{1 - ae^{-j3\omega}} \right) = \left( \frac{1 - e^{-j3\omega}}{j\omega} \right) \left( \frac{1}{1 - ae^{-j3\omega}} \right)
\]

Alternatively, apply the formula directly and simplify:

\[
X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \\
= \int_0^3 a^0 e^{-j\omega t} dt + \int_3^6 a^1 e^{-j\omega t} dt + \int_6^9 a^2 e^{-j\omega t} dt + \ldots \\
= a^0 \left[ \frac{e^{-j\omega t}}{-j\omega} \right]_0^3 + a^1 \left[ \frac{e^{-j\omega t}}{-j\omega} \right]_3^6 + a^2 \left[ \frac{e^{-j\omega t}}{-j\omega} \right]_6^9 + \ldots \\
= -\frac{1}{j\omega} \left( a^0 \left( e^{-j\omega 3} - e^{-j\omega 0} \right) + a^1 e^{-j\omega 3} \left( e^{-j\omega 3} - e^{-j\omega 0} \right) + a^2 e^{-j\omega 6} \left( e^{-j\omega 3} - e^{-j\omega 0} \right) \right) \\
= \left( \frac{1 - e^{-j3\omega}}{j\omega} \right) \left( \sum_{m=0}^{\infty} (ae^{-j3\omega})^m \right) \\
= \left( \frac{1 - e^{-j3\omega}}{j\omega} \right) \left( \frac{1}{1 - ae^{-j3\omega}} \right)
\]
9 Signal Facts (10 Points)

Suppose we are given the following facts about a signal $x[\cdot]$ (and its Fourier series coefficients $X[\cdot]$):

1. $x[\cdot]$ is periodic and real-valued.
2. $\max(x[n]) = 5$
3. $X[k - 6] = X[k]$ for all $k$
4. $X[2] = 0$
5. $y[n] = x[n] - 1$ is a antisymmetric function of $n$.
6. $\frac{1}{6} \sum_{n=-2}^{3} (-1)^n x[n] = 0$
7. $\text{Im}(X[1]) > 0$
Fact 1 tells us that our Fourier series must be conjugate symmetric, i.e., that \( X[k] = X^*[−k] \) for all \( k \).

Fact 3 tells us that the \( x[n] \) is periodic in \( N = 6 \) (though this alone does not tell us that \( N = 6 \) is the fundamental period of the signal).

Fact 4 tells us that \( X[2] = 0 \), which, by the conjugate symmetry, tells us that \( X[−2] = X[4] \) is also 0.

In order for fact 5 to hold, \( X[0] \) must be 1 (\( y[n] \) being antisymmetric means that its DC component must be 0, since \( x[n] = y[n] + 1 \), \( x[·]'s \) DC component must be 1.

Fact 6 tells us that \( X[3] = 0 \), since:

\[
\frac{1}{6} \sum_{n=-2}^{3} (-1)^n x[n] = \frac{1}{6} \sum_{n=-2}^{3} (e^{-j\pi n})x[n] = \frac{1}{6} \sum_{n=-2}^{3} (e^{-j\frac{2\pi n}{6}})x[n] = X[3]
\]

The only remaining pieces to figure out are \( X[1] \) and \( X[−1] = X[5] \). Since the function with its DC component removed is purely antisymmetric, we know that \( X[1] \) and \( X[5] \) must be purely imaginary. Combining this with fact 7, we find that \( X[1] = mj \) and \( X[5] = −mj \) for some value of \( m \).

Putting all these together, we find that \( x[n] = 1 − 2m \sin\left(\frac{\pi}{3}n\right) \).

Fact 2 tells us that the maximum value \( x[·] \) reaches is 5. Thus, we need \( m \) such that \( 2m \times \max_n \left(\sin\left(\frac{\pi}{3}n\right)\right) = 5 \)

The values that \( \sin\left(\frac{\pi}{3}n\right) \) takes in one period are:

\[
\sin(0) = 0 \quad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \quad \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2} \quad \sin(\pi) = 0 \\
\sin\left(\frac{4\pi}{3}\right) = -\sin\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2} \quad \sin\left(\frac{5\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}
\]

Using this and solving, we find that \( m = \frac{4}{\sqrt{3}} \), which gives the answers on the following page.

Answer the following questions about this signal. If there is not enough information to solve any of the questions, enter NEI (for “not enough information”) in those boxes.

What is this signal’s fundamental period? \( N = \boxed{6} \)

What are the values of \( X[·] \)?
\[ X[k] = X[k + 6] = \begin{cases} 
1 & \text{if } k = 0 \\
4j/\sqrt{3} & \text{if } k = 1 \\
-4j/\sqrt{3} & \text{if } k = -1 \\
0 & \text{otherwise} 
\end{cases} \]

Determine a closed-form expression for \( x[n] \) that does not include complex exponential functions:

\[
x[n] = 1 - \frac{8}{\sqrt{3}} \sin \left( \frac{\pi}{3} n \right) \]