Quiz 2 Review Problems

Problem 1: Echo
Problem 2: Slow down

Quiz 2: Tuesday, April 21.

- Two hours – sign up for your preferred time.
- Coverage up to and including all of week 10.
- Closed book except for two pages of notes (four sides total).
- No electronic devices.
Problem 1: Echo

Assume that a single echo interferes with a speaker’s voice that is being recorded by a microphone as illustrated in the following figure.

We wish to characterize the effect of this echo on the speaker’s voice.
Part 1

We can represent this recording situation as a linear, time-invariant system, with the speaker’s voice as the input and the recorded microphone signal as the output.

\[ x(t) \rightarrow h(t) \rightarrow y(t) \]

Assume that the impulse response of this system is

\[ h(t) = \delta(t-T_1) + \epsilon \delta(t-T_2) \]

where \( T_1 \) represents the delay of the direct path from speaker to microphone, \( T_2 \) represents the delay through the echo path, and \( \epsilon \) represents the amplitude of the echo.

Sketch the magnitude and angle of the frequency response of this system in the absence of an echo (i.e., when \( \epsilon = 0 \)).
Part 1

In the absence of an echo (i.e., when $\epsilon = 0$), the impulse response simplifies.

$$h(t) = \delta(t-T_1) + \epsilon \delta(t-T_2) \xrightarrow{\epsilon=0} \delta(t-T_1)$$

The frequency response of this system is the DTFT of $h(t) = \delta(t - T_1)$.

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t} \, dt$$

$$= \int_{-\infty}^{\infty} \delta(t-T_1)e^{-j\omega t} \, dt$$

$$= e^{-j\omega T_1}$$

The magnitude of $H(\omega)$ is 1 and the angle is $-\omega T_1$. 
Part 1

Sketch the magnitude and angle of the frequency response of this system in the absence of an echo (i.e., when $\epsilon = 0$). Label all important values.

$$H(\omega) = e^{-j\omega T_1}$$

The magnitude of $H(\omega)$ is 1 and the angle is $-\omega T_1$.

![Magnitude and Angle Diagram](image)
Check Yourself

It makes sense that the magnitude is 1 for all frequencies. Sinusoidal signals are not amplified or attenuated based on frequency.

But why does the angle vary linearly with $\omega$?

The same amount of time corresponds to different amounts of phase.

To get the same time delay ($T_1$) we need more phase at higher frequencies.
Part 2

Next consider the case when there is an echo. The following plots show the magnitude and angle of the frequency response of this system for frequencies $-1500 < \omega < 1500 \text{ rad/s}$, for unknown values of $T_1, T_2,$ and $\epsilon$.

Determine values of $T_1, T_2,$ and $\epsilon$ that are consistent with the plots above.
Part 2

Start with the impulse response.

\[ h(t) = \delta(t - T_1) + \epsilon \delta(t - T_2) \]

Take the Fourier transform of \( h(t) \) to obtain the frequency response.

\[ H(\omega) = e^{-j\omega T_1} + \epsilon e^{-j\omega T_2} = e^{-j\omega T_1}(1 + \epsilon e^{-j\omega(T_2 - T_1)}) \]

Magnitude:

\[
|H(\omega)| = \left| e^{-j\omega T_1} \right| \times \left| 1 + \epsilon e^{-j\omega(T_2 - T_1)} \right| \\
= \sqrt{\left(1 + \epsilon \cos \omega(T_2 - T_1)\right)^2 + \epsilon^2 \sin^2 \omega(T_2 - T_1)} \\
= \sqrt{1 + 2\epsilon \cos \omega(T_2 - T_1) + \epsilon^2}
\]

Angle:

\[ \angle H(\omega) = -\omega T_1 + \angle \left(1 + \epsilon e^{-j\omega(T_2 - T_1)}\right) \]
Part 2
Relate the expressions for magnitude and angle to the plots.

\[ |H(\omega)| = \sqrt{1 + 2\epsilon \cos \omega(T_2 - T_1) + \epsilon^2} \]
\[ \angle H(\omega) = -\omega T_1 + \angle \left( 1 + \epsilon e^{-j\omega(T_2-T_1)} \right) \]

The angle oscillates about a straight line determined by the first term.

\[ \angle H(\omega) = -\omega T_1 + \angle \left( 1 + \epsilon e^{-j\omega(T_2-T_1)} \right) \]

From the plot, the average slope of the phase curve is \(-\frac{\pi}{1500}\). From the equation, the average slope is \(- T_1\).

Therefore \( T_1 = \frac{\pi}{1500} \).

The magnitude oscillates between \(1+\epsilon\) and \(1-\epsilon\) with a period of \( \omega = \frac{2\pi}{(T_2-T_1)} \). From the magnitude plot, we can see that \( \epsilon \approx 0.2 \) and \( \frac{2\pi}{(T_2-T_1)} \approx 1500/2 \), so that \( T_2-T_1 \approx \frac{4\pi}{1500} \).

Then \( T_2 \approx \frac{4\pi}{1500} + \frac{\pi}{1500} = \frac{\pi}{300} \).
Relate the expressions for magnitude and angle to the plots.
Check Yourself

Start with the case when $\epsilon = 0$.
Then $H(\omega) = e^{-j\omega T_1}$.

We can think of this as a unit vector in the complex plane.
The vector rotates clockwise as $\omega$ increases.

\[
\begin{align*}
|H(\omega)| & \quad \omega \\
-1500 & \quad 0 & \quad 1500
\end{align*}
\]

\[
\begin{align*}
\angle H(\omega) & \quad \omega \\
-\pi & \quad 0 & \quad \pi
\end{align*}
\]

slope $= -T_1$
Check Yourself

When $\epsilon \neq 0$ there are two vectors, one corresponding to each term in $H(\omega)$.

$$H(\omega) = e^{-j\omega T_1} + \epsilon e^{-j\omega T_2}$$

The first is the same as it was for $\epsilon = 0$.
The second has length $\epsilon$ and rotates with speed proportional to $T_2$.
$T_2 = \pi/300$ is bigger than $T_1 = \pi/1500$ so the small vector rotates faster.

Results from this vector diagram matches our previous math.  

\[\sqrt{\text{ }}\]
Problem 2: Slow Down

Let $x[n]$ represent a discrete time signal whose DTFT is given by

$$X(\Omega) = \begin{cases} 1 & \text{if } |\Omega| < \frac{\pi}{5} \\ 0 & \text{if } \frac{\pi}{5} < |\Omega| < \pi \end{cases}$$

and is periodic in $\Omega = 2\pi$.

![Diagram of DTFT of $X(\Omega)$]

A new signal $y_0[n]$ is derived by stretching $x[n]$ as follows:

$$y_0[n] = \begin{cases} x[n/2] & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

![Diagram of $x[n]$ and $y_0[n]$]

Part 1. Sketch the magnitude and angle of $Y_0(\Omega)$, the DTFT of $y_0[n]$. 
Part 1

Sketch the magnitude and angle of $Y_0(\Omega)$, which is the DTFT of $y_0[n]$.

$$y_0[n] = \begin{cases} x[n/2] & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\Omega n} = X(2\Omega)$$
Part 1

Sketch the magnitude and angle of $Y_0(\Omega)$, which is the DTFT of $y_0[n]$.

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$Y(\Omega) = X(2\Omega)$$

Notice that $X(\Omega)$ was lowpass, but $Y(\Omega)$ has both lowpass and highpass regions. Where did the highpass regions come from?
Part 2a

The $y_0[n]$ signal alternates between original values of $x[n]$ and zeros. This alternation creates high frequencies that were not found in $x[n]$. To reduce these high frequencies, we can convolve $y_0[n]$ with $h_1[n]$ given by

$$h_1[n] = \delta[n] + \delta[n-1]$$

to generate a new signal

$$y_1[n] = (y_0 \ast h_1)[n]$$

Sketch the first 10 samples of $y_1[n]$. Briefly describe the effect of this convolution.
Part 2a

The $y_0[n]$ signal alternates between original values of $x[n]$ and zeros. This alternation creates high frequencies that were not found in $x[n]$. To reduce these high frequencies, we can convolve $y_0[n]$ with $h_1[n]$ given by

$$h_1[n] = \delta[n] + \delta[n-1]$$

to generate a new signal

$$y_1[n] = (y_0 * h_1)[n]$$

Sketch the first 10 samples of $y_1[n]$.

Briefly describe the effect of this convolution.

\[ y_1[n] \]

0 10 n

Effect: Each zero-value that was inserted into $y_0[n]$ has been replaced by a copy of the previous (non-zero) sample ($y_0[n-1]$).
Part 2b

Let $H_1(\Omega)$ represent the DTFT of $h_1[n]$.
Sketch the magnitude and angle of $H_1(\Omega)$. 
Part 2b

Let \( H_1(\Omega) \) represent the DTFT of \( h_1[n] \).

Sketch the magnitude and angle of \( H_1(\Omega) \).

\[
H_1(\Omega) = \sum_{n=-\infty}^{\infty} h_1[n] e^{-j\Omega n} = 1 + e^{-j\Omega} = e^{-j\Omega/2} \left( e^{j\Omega/2} + e^{-j\Omega/2} \right)
\]

\[
= 2 e^{-j\Omega/2} \cos(\Omega/2)
\]

The effect of convolving \( y_0[n] \) with \( h_1[n] \) is to filter \( Y_0(\Omega) \) by \( H_1(\Omega) \).

Briefly describe the effect of this filtering on \( Y_0(\Omega) \).

Effect: When \( \Omega \) is near \( \pi \), the magnitude of \( H_1(\Omega) \) is small. Thus \( H_1(\Omega) \) reduces the high frequency components that were introduced by inserting zeros in \( x[n] \).
Part 2b

\[
|Y_0(\Omega)| = |H_1(\Omega)| \times |Y_0(\Omega)|
\]
Part 3a

An even better way to smooth $y_0[n]$ is to convolve it with $h_2[n]$ given by

$$h_2[n] = \frac{1}{2} \delta[n + 1] + \delta[n] + \frac{1}{2} \delta[n - 1]$$

to generate a new signal

$$y_2[n] = (y_0 * h_2)[n].$$

Sketch the first 10 samples of $y_2[n]$

Briefly describe the relation between $y_0[n]$ and $y_2[n]$. 
Part 3a

An even better way to smooth $y_0[n]$ is to convolve it with $h_2[n]$ given by

$$h_2[n] = \frac{1}{2}\delta[n + 1] + \delta[n] + \frac{1}{2}\delta[n - 1]$$

to generate a new signal

$$y_2[n] = (y_0 \ast h_2)[n].$$

Sketch the first 10 samples of $y_2[n]$

Briefly describe the relation between $y_0[n]$ and $y_2[n]$.

Relation: Each zero-value that was inserted into $y_0[n]$ has been replaced with the average of the samples on either side of it.
Part 3b

Let $H_2(\Omega)$ represent the DTFT of $h_2[n]$. Sketch the magnitude and angle of $H_2(\Omega)$.

$$H_2(\Omega) = \sum_{n=-\infty}^{\infty} h_2[n] e^{-j\Omega n} = 1 + \frac{1}{2} e^{-j\Omega} + \frac{1}{2} e^{j\Omega} = 1 + \cos \Omega$$

When $\Omega$ is near $\pi$, the magnitude of $H_2(\Omega)$ is small. Thus $H_2(\Omega)$ reduces the high frequency components that were introduced by inserting zeros in $x[n]$. However, $H_2(\Omega)$ introduces no phase shift, while $H_1(\Omega)$ introduced a delay. Also, the amplitudes of frequency components near $\pi$ are reduced more by $H_2$ than by $H_1$. 
Part 3b

\[ |Y_0(\Omega)| \]

\[ |H_2(\Omega)| \]

\[ |Y_2(\Omega)| = |H_2(\Omega)| \times |Y_0(\Omega)| \]
Summary
We worked through two sample quiz problems.
Problem 1: Echo
Problem 2: Slow down

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