These slides repeat those for the lecture on March 12 (lec06b), which was optional.
The Signals and System Abstraction

Describe a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.

This is particularly useful for systems that are **linear and time-invariant**.

Such systems can be analyzed using three different representations.

- **Difference Equation:** algebraic **constraint** on samples
- **Convolution:** represent a system by its **unit-sample response**
- **Filter:** represent a system by its **frequency response**
Linear Difference Equations with Constant Coefficients

Discrete-time systems that are linear and time-invariant can be **completely specified** by a linear difference equation with constant coefficients.

**General form:**

\[ \sum_{l} c_{l}y[n-l] = \sum_{m} d_{m}x[n-m] \]

Such systems are easily shown to be linear and time-invariant.

**Additivity:** output of sum is sum of outputs

\[ \sum_{l} c_{l}(y_{1}[n-l] + y_{2}[n-l]) = \sum_{m} d_{m}(x_{1}[n-m] + x_{2}[n-m]) \]

**Homogeneity:** scaling an input scales its output

\[ \sum_{l} \alpha c_{l}y_{1}[n-l] = \sum_{m} \alpha d_{m}x_{1}[n-m] \]

**Time invariance:** delaying an input delays its output

\[ \sum_{l} c_{l}y_{1}[(n-n_{0})-l] = \sum_{m} d_{m}x_{1}[(n-n_{0})-m] \]
Unit-Sample Response

If a system is linear and time-invariant (LTI), its input-output relation is **completely specified** by the system’s unit-sample response $h[n]$.

1. One can always find the unit-sample response of a system.
   \[ \delta[n] \xrightarrow{\text{LTI}} h[n] \]

2. Time invariance implies that shifting the input simply shifts the output.
   \[ \delta[n - k] \xrightarrow{\text{LTI}} h[n - k] \]

3. Homogeneity implies that scaling the input simply scales the output.
   \[ x[k] \delta[n - k] \xrightarrow{\text{LTI}} x[k] h[n - k] \]

4. Additivity implies that the response to a sum is the sum of responses.
   \[ x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - k] \xrightarrow{\text{LTI}} y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n - k] \equiv (x \ast h)[n] \]

The output of an LTI system can **always** be found by convolving: $(x \ast h)[n]$. 
**Unit-Sample Response**

The unit-sample response is a **complete** description of a system.

$$\delta[n] \rightarrow \text{LTI} \rightarrow h[n]$$

This is a bit surprising since $\delta[n]$ is such a **simple signal**. The unit-sample signal is the **shortest** possible non-trivial DT signal!

The response to this simple signal can be used to determine the response to **any** other input.
Frequency Response

The frequency response is a third way to characterize a linear time-invariant system. This characterization is based on responses to sinusoids.

\[
\cos(\Omega n) \quad \overset{\text{LTI}}{\longrightarrow} \quad A \cos(\Omega n - \phi)
\]

The idea is to characterize a system by the way \(A\) and \(\phi\) vary with \(\Omega\).

Sinusoids differ from the unit-sample signal in important ways:
- eternal (longest possible signals) versus transient (shortest possible)
- comprises a single frequency versus a sum of all possible frequencies
**Frequency Response**

Using complex exponentials to characterize the frequency response.

\[ e^{j\Omega n} \rightarrow \text{LTI} \rightarrow Ae^{j\Omega n} \]

Notice that the complex valued \( A \) can represent both amplitude and phase. We can find \( A \) using convolution.

\[
y[n] = (x \ast h)[n] = \sum_{m=-\infty}^{\infty} x[n - m]h[m] = \sum_{m=-\infty}^{\infty} e^{j\Omega(n-m)}h[m] \\
= e^{j\Omega n} \sum_{m=-\infty}^{\infty} h[m]e^{-j\Omega m} = H(\Omega) e^{j\Omega n}
\]

The response to a complex exponential is a complex exponential with the **same frequency** but possibly **different amplitude and phase**.

The map for how a system modifies the amplitude and phase of a complex exponential input is the **Fourier transform of the unit-sample response**.
Frequency Response

The frequency response is a **complete** characterization of an LTI system.

1. One can always find the frequency response of a system.
   \[
e^{j\Omega n} \quad \xrightarrow{\text{LTI}} \quad H(\Omega)e^{j\Omega n}
   \]

2. Scaling the input by a constant scales the output by the same constant.
   \[
   X(\Omega)e^{j\Omega n} \quad \xrightarrow{\text{LTI}} \quad X(\Omega)H(\Omega)e^{j\Omega n}
   \]

3. Linearity implies that the response to a sum is the sum of the responses.
   \[
   \frac{1}{2\pi} \int_{2\pi} X(\Omega)e^{j\Omega n} d\Omega \quad \xrightarrow{\text{LTI}} \quad \frac{1}{2\pi} \int_{2\pi} X(\Omega)H(\Omega)e^{j\Omega n} d\Omega
   \]

4. The Fourier transform of the output is \(X(\Omega)H(\Omega)\).
   \[
   X(\Omega) \quad \xrightarrow{\text{LTI}} \quad X(\Omega)H(\Omega)
   \]

The output transform can **always** be found by multiplying \(X(\Omega)\) by \(H(\Omega)\).
Frequency Response

The frequency response can be the most insightful description of a system.

Example:
A low-pass filter passes frequencies near 0 and rejects those near $\pi$.

$$H(\Omega)$$

Very natural way to describe audio enhancements:
- bass-boost
- room equalizer
- tone control
System Abstraction

Two complete representations for linear, time-invariant systems.

Unit-Sample Response: responses across time for a unit-sample input.

Frequency Response: responses across frequencies for sinusoidal inputs.

The frequency response is Fourier transform of unit-sample response!
Example

Find the frequency response of a system described by the following difference equation.

\[ y[n] - \alpha y[n-1] = x[n] \]

**Method 1:**
Find the unit-sample response and take its Fourier transform.

\[ x[n] = \delta[n] \]

Solve the difference equation for \( y[n] \).

\[ y[n] = x[n] + \alpha y[n-1] \]

If we knew \( y[-1] \), we could use this equation to find \( y[n] \) for \( n \geq 0 \).

Assume that \( y[n] = 0 \) if \( n < 0 \) (initial values).

\[
\begin{align*}
y[0] &= \delta[0] + \alpha y[-1] = 1 \\
y[1] &= \delta[1] + \alpha y[0] = \alpha \\
\end{align*}
\]

\[ h[n] = \alpha^n u[n] = \begin{cases} 
\alpha^n & \text{if } n \geq 0 \\
0 & \text{otherwise}
\end{cases} \]
Example

The frequency response is the Fourier transform of $h[n]$.

\[ h[n] = \alpha^n u[n] = \begin{cases} \alpha^n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases} \]

\[
H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\Omega n} = \sum_{n=0}^{\infty} \left(\alpha e^{-j\Omega}\right)^n = \frac{1}{1-\alpha e^{-j\Omega}}
\]
Example

Find the frequency response of a system described by the following difference equation.

\[ y[n] - \alpha y[n-1] = x[n] \]

**Method 2:**

Find the response to \( e^{j\Omega n} \) directly.

\[ x[n] = e^{j\Omega n} \]

Because the system is linear and time-invariant, the output will have the same frequency as the input, but possibly different amplitude and phase.

\[ y[n] = H(\Omega)e^{j\Omega n} \]

\[ y[n-1] = H(\Omega)e^{j\Omega(n-1)} = H(\Omega)e^{-j\Omega}e^{j\Omega n} \]

Substitute into the difference equation.

\[ H(\Omega)e^{j\Omega n} - \alpha H(\Omega)e^{-j\Omega}e^{j\Omega n} = H(\Omega)(1-\alpha e^{-j\Omega})e^{j\Omega n} = e^{j\Omega n} \]

Since \( e^{j\Omega n} \) is never 0, we can divide it out.

\[ H(\Omega) = \frac{1}{1 - \alpha e^{-j\Omega}} \]

Same answer as method 1.
Example

Find the frequency response of a system described by the following difference equation.

\[ y[n] - \alpha y[n-1] = x[n] \]

**Method 3:**

Take the Fourier transform of the difference equation.

\[ Y(\Omega) - \alpha e^{-j\Omega} Y(\Omega) = X(\Omega) \]

Solve for \( Y(\Omega) \).

\[ Y(\Omega) = \frac{1}{1 - \alpha e^{-j\Omega}} X(\Omega) \]

Since \( Y(\Omega) = H(\Omega)X(\Omega) \),

\[ H(\Omega) = \frac{1}{1 - \alpha e^{-j\Omega}} \]

Same answer as methods 1 and 2.
Example
Plot the frequency response.

\[ H(\Omega) = \frac{1}{1 - \alpha e^{-j\Omega}} \]

Note that denominator is sum of 2 complex numbers.

0 < \alpha < 1:

Amplifies low frequencies, attenuates high frequencies, adds phase delay.
The Signals and System Abstraction

This abstraction applies equally well for continuous-time signals.
Describe a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.

This is particularly useful for systems that are **linear and time-invariant**.

Such systems can be analyzed using three different representations.

- **Differential Equation**: algebraic **constraint** on derivatives.
- **Convolution**: represent a system by its **unit-impulse response**
- **Filter**: represent a system by its **frequency response**
Linear Differential Equations with Constant Coefficients

If a continuous-time system can be described by a linear differential equation with constant coefficients, then the system is linear and time-invariant.

**General form:**

$$\sum_{l} c_l \frac{d^l}{dt^l} y(t) = \sum_{m} d_m \frac{d^m}{dt^m} x(t)$$

**Additivity:** output of sum is sum of outputs

$$\sum_{l} c_l \frac{d^l}{dt^l} \left( y_1(t) + y_2(t) \right) = \sum_{m} d_m \frac{d^m}{dt^m} \left( x_1(t) + x_2(t) \right)$$

**Homogeneity:** scaling an input scales its output

$$\sum_{l} c_l \frac{d^l}{dt^l} \left( \alpha y(t) \right) = \sum_{m} d_m \frac{d^m}{dt^m} \left( \alpha x(t) \right)$$

**Time invariance:** delaying an input delays its output

$$\sum_{l} c_l \frac{d^l}{dt^l} y(t-t_0) = \sum_{m} d_m \frac{d^m}{dt^m} x(t-t_0)$$
Impulse Response

A CT system is completely characterized by its impulse response, much as a DT system is completely characterized by its unit-sample response.

We have worked with the impulse (Dirac delta) function $\delta(t)$ previously. It’s defined in a limit as follows.

Let $p_\Delta(t)$ represent a pulse of width $\Delta$ and height $\frac{1}{\Delta}$ so that its area is 1.

Then

$$\delta(t) = \lim_{\Delta \to 0} p_\Delta(t)$$

The impulse function can be used to break an arbitrary input $x(t)$ into time-based components, much as $\delta[k]$ is used for discrete-time signals.
**Impulse Response**

An arbitrary CT signal can be represented by an infinite sum of infinitesimal impulses (which define an integral).

Approximate an arbitrary signal $x(t)$ (blue) as a sum of pulses $p_\Delta(t)$ (red).

$$x_\Delta(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)p_\Delta(t - k\Delta)\Delta$$

and the limit of $x_\Delta(t)$ as $\Delta \to 0$ will approximate $x(t)$.

$$\lim_{\Delta \to 0} x_\Delta(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)p_\Delta(t - k\Delta)\Delta \to \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)\,d\tau$$

The result in CT is much like the result for DT:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)\,d\tau \quad x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta(n - m)$$
**Impulse Response**

If a system is linear and time-invariant (LTI), its input-output relation is **completely specified** by the system’s impulse response $h(t)$.

1. One can always find the impulse response of a system.
   \[
   \delta(t) \xrightarrow{\text{system}} h(t)
   \]

2. Time invariance implies that shifting the input simply shifts the output.
   \[
   \delta(t-\tau) \xrightarrow{\text{system}} h(t-\tau)
   \]

3. Homogeneity implies that scaling the input simply scales the output.
   \[
   x(\tau)\delta(t-\tau) \xrightarrow{\text{system}} x(\tau)h(t-\tau)
   \]

4. Additivity implies that the response to a sum is the sum of responses.
   \[
   x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau \xrightarrow{\text{system}} y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \equiv (x \ast h)(t)
   \]

The output of an LTI system can **always** be found by convolving: $(x \ast h)(t)$. 
Impulse Response

The impulse response is a complete description of a system.

\[ \delta(t) \rightarrow \text{LTI} \rightarrow h(t) \]

This is a bit surprising since \( \delta(t) \) is zero almost everywhere. The impulse function is the shortest possible non-trivial CT signal!

The response to this signal can be used to determine the response to any other input.
Frequency Response

A frequency response is a third way to characterize a linear time-invariant system. This characterization is based on responses to sinusoids.

\[ \cos(\omega t) \rightarrow \text{LTI} \rightarrow A \cos(\omega t - \phi) \]

The idea is to characterize a system by the way \( A \) and \( \phi \) vary with \( \omega \).

Sinusoids differ from the unit-sample signal in important ways:
- eternal (longest possible signals) versus transient (shortest possible)
- comprises a single frequency versus a sum of all possible frequencies
**Frequency Response**

Using complex exponentials to characterize the frequency response.

\[ e^{j\omega t} \quad \xrightarrow{\text{LTI}} \quad Ae^{j\omega t} \]

Notice that the complex valued \( A \) can represent both amplitude and phase. We can find \( A \) using convolution.

\[
y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau) = \int_{-\infty}^{\infty} e^{j\omega(t-\tau)}h(\tau) \\
= e^{j\omega t} \int_{-\infty}^{\infty} h(\tau)e^{-j\omega \tau} = H(\omega) e^{j\omega t}
\]

The response to a complex exponential is a complex exponential with the **same frequency** but possibly **different amplitude and phase**.

The map for how a system modifies the amplitude and phase of a complex exponential input is the **Fourier transform of the impulse response**.
**Frequency Response**

The frequency response is a complete characterization of an LTI system.

1. One can always find the frequency response of a system.

\[
e^{j\omega t} \xrightarrow{\text{system}} H(\omega) e^{j\omega t}
\]

2. Scaling the input by a constant scales the output by the same constant.

\[
X(\omega) e^{j\omega t} \xrightarrow{\text{system}} X(\omega) H(\omega) e^{j\omega t}
\]

3. Linearity implies that the response to a sum is the sum of the responses.

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \xrightarrow{\text{system}} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) H(\omega) e^{j\omega t} d\omega
\]

4. The Fourier transform of the output is \(X(\omega) H(\omega)\).

\[
X(\omega) \xrightarrow{\text{system}} X(\omega) H(\omega)
\]

The output transform can always be found by multiplying \(X(\omega)\) by \(H(\omega)\).
Frequency Response

The frequency response can be the most insightful description of a system.

Example:
A low-pass filter passes frequencies near 0 and rejects those near $\pi$.

Very natural way to describe audio enhancements:
- bass-boost
- room equalizer
- tone control
System Abstraction

Two complete representations for linear, time-invariant systems.

**Impulse Response:** responses across time for a impulse input.

**Frequency Response:** responses across frequencies for sinusoidal inputs.

The frequency response is Fourier transform of impulse response!
Example

Find the frequency response of a system described by the following differential equation.

\[ y(t) + \alpha \frac{dy(t)}{dt} = x(t) \]

**Method 1:**

Find the response to \( e^{j\omega t} \) directly.

\[ x(t) = e^{j\omega t} \]

Because the system is linear and time-invariant, the output will have the same frequency as the input, but possibly different amplitude and phase.

\[ y(t) = H(\omega)e^{j\omega t} \]

\[ \frac{dy(t)}{dt} = j\omega H(\omega)e^{j\omega t} \]

Substitute into the differential equation.

\[ H(\omega)e^{j\omega t} + j\omega \alpha H(\omega)e^{j\omega t} = (1 + j\omega \alpha)H(\omega)e^{j\omega t} = e^{j\omega t} \]

Since \( e^{j\omega t} \) is never 0, we can divide it out.

\[ H(\omega) = \frac{1}{1 + j\omega \alpha} \]
Example

Find the frequency response of a system described by the following differential equation.

\[ y(t) + \alpha \frac{dy(t)}{dt} = x(t) \]

**Method 2:**

Take the Fourier transform of the differential equation.

\[ Y(\omega) + j\omega \alpha Y(\omega) = X(\omega) \]

Solve for \( Y(\omega) \).

\[ Y(\omega) = \frac{1}{1 + j\omega \alpha} X(\omega) \]

Since \( Y(\omega) = H(\omega) X(\omega) \),

\[ H(\omega) = \frac{1}{1 + j\omega \alpha} \]

Same answer as method 1.
Example

Plot the frequency response.

\[ H(\omega) = \frac{1}{1 + j\omega \alpha} \]

Note that denominator is sum of 2 complex numbers.

Amplifies low frequencies, attenuates high frequencies, adds phase delay.
System Abstraction

Two complete representations for linear, time-invariant systems.

Impulse Response: responses across time for a unit-sample input.

Frequency Response: responses across frequencies for sinusoidal inputs.

The frequency response is Fourier transform of unit-sample response!