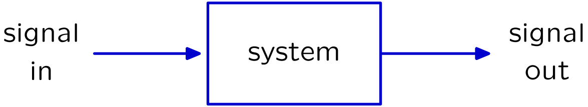
6.003: Signal Processing

Impulse Response and Convolution

March 10, 2020

The Signals and System Abstraction

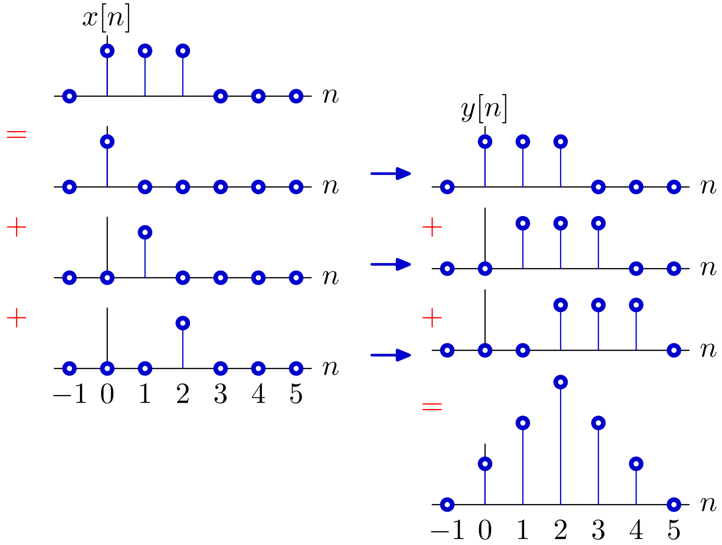
Describe a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.



This is particularly useful for systems that are **linear and time-invariant**.

Superposition

Break input into additive parts and sum the responses to the parts.



Superposition

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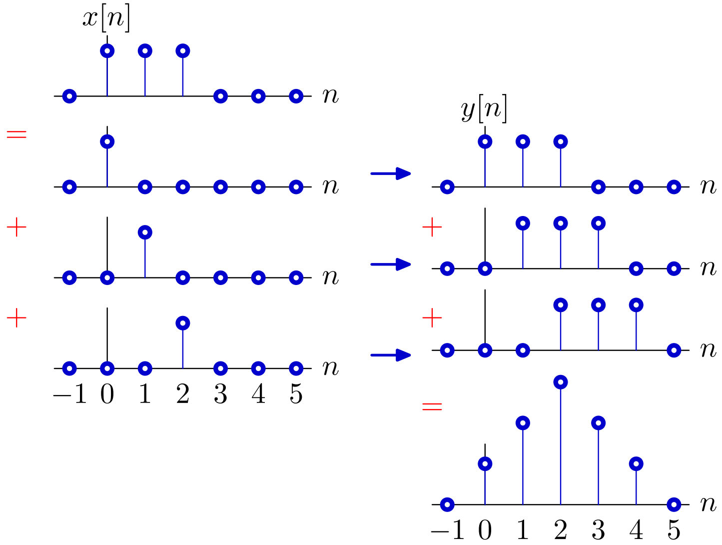
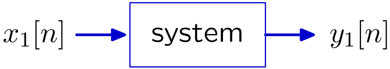


Superposition works because the system is **linear**.

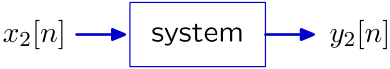
Linearity

A system is linear if its response to a weighted sum of inputs is equal to the weighted sum of its responses to each of the inputs.

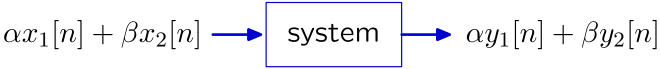
Given



and



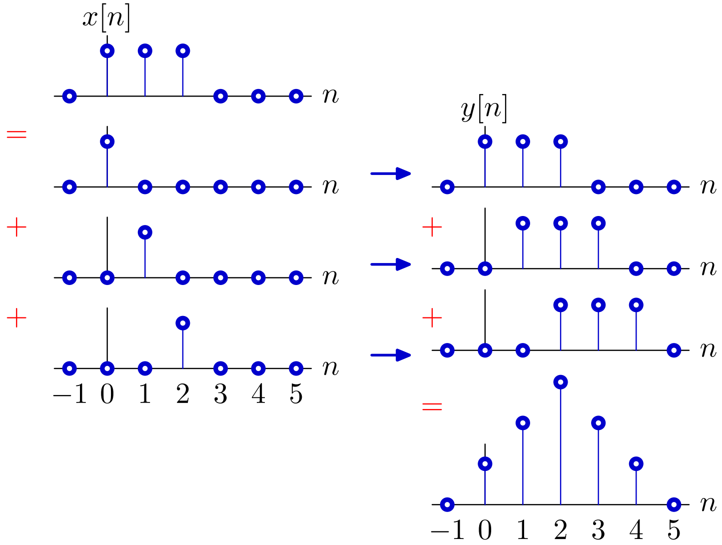
the system is linear if



is true for all α and β and all times n.

Superposition

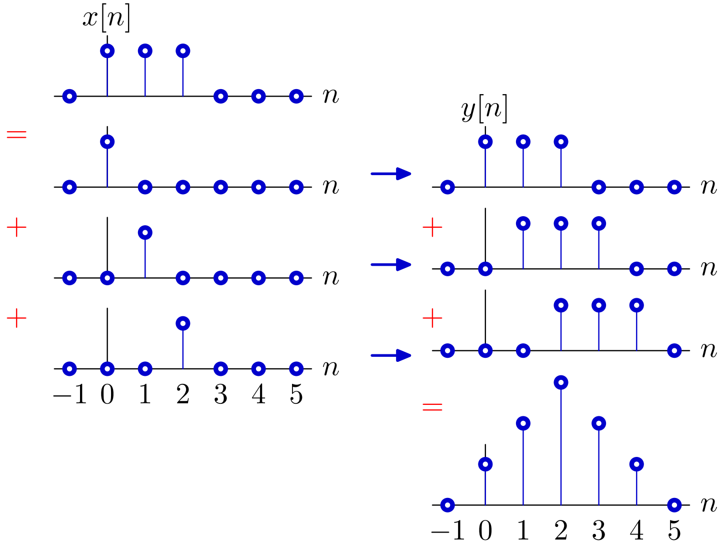
Break input into additive parts and sum the responses to the parts.



Superposition works if the system is **linear**.

Superposition

Break input into additive parts and sum the responses to the parts.

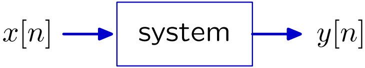


Reponses to parts are easy to compute if system is **time-invariant**.

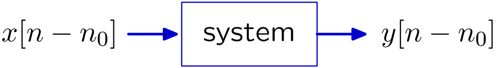
Time-Invariance

A system is time-invariant if delaying the input to the system simply delays the output by the same amount of time.

Given



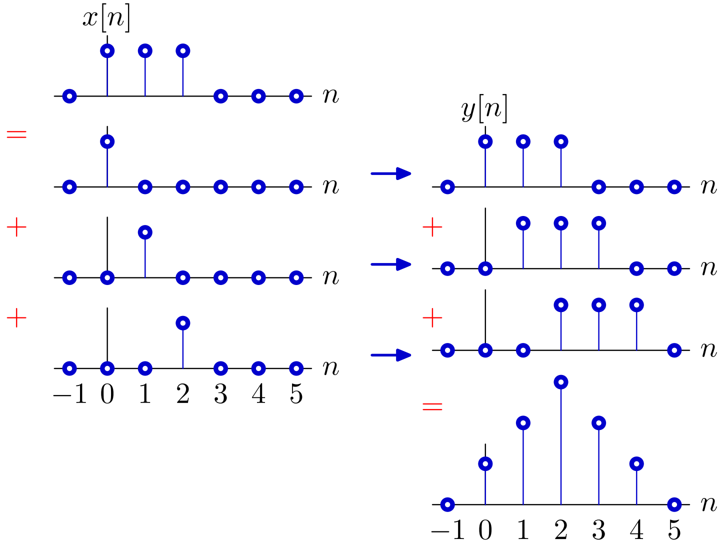
the system is time invariant if



is true for all n and all n_0.

Superposition

Break input into additive parts and sum the responses to the parts.



Superposition is easy if the system is **linear** and **time-invariant**.

Unit-Sample Response

If a system is linear and time-invariant (LTI), its input-output relation is **completely specified** by the system's unit-sample response $h[n]$.

1. One can always find the unit-sample response of a system.

$$\delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n]$$

2. Time invariance implies that shifting the input simply shifts the output.

$$\delta[n-k] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n-k]$$

3. Homogeneity implies that scaling the input simply scales the output.

$$x[k]\delta[n-k] \longrightarrow \boxed{\text{LTI}} \longrightarrow x[k]h[n-k]$$

4. Additivity implies that the response to a sum is the sum of responses.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv (x*h)[n]$$

The output of an LTI system can **always** be found by convolving: $(x*h)[n]$.

Convolution

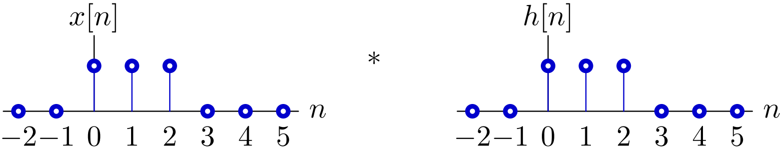
Response of an LTI system to an arbitrary input.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv (x * h)[n]$$

This operation is called **convolution**.

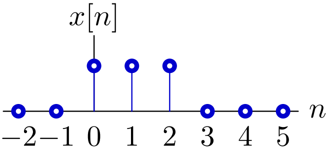
Structure of Convolution

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$



Structure of Convolution

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k]$$



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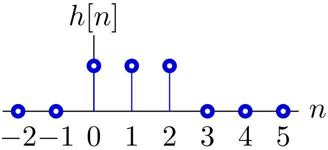

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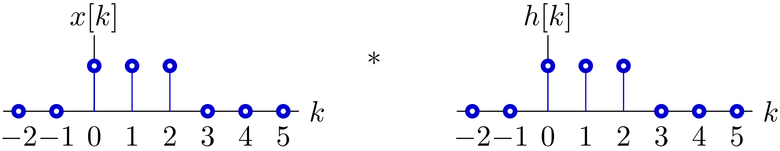

Structure of Convolution

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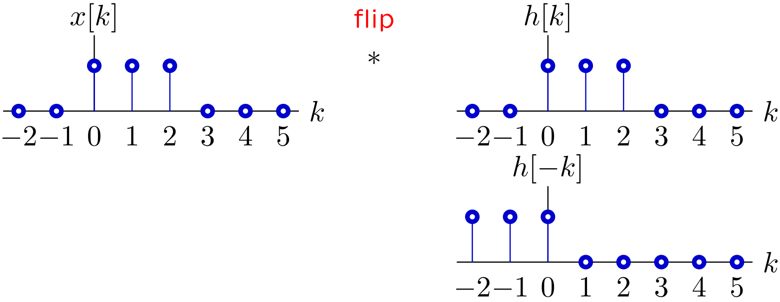

Structure of Convolution

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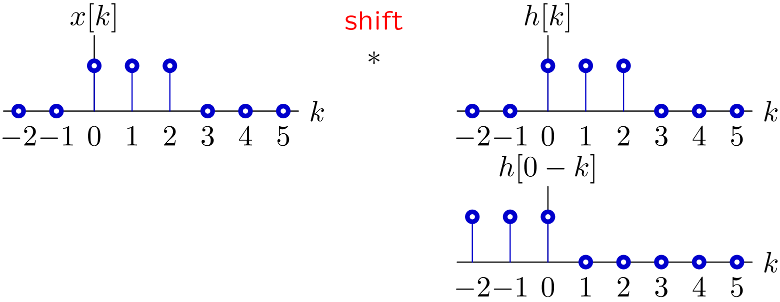

Structure of Convolution

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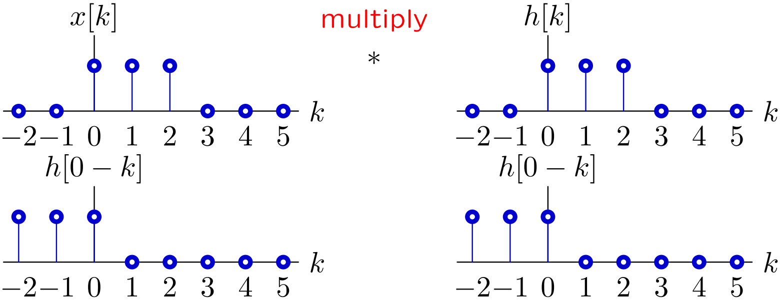

Structure of Convolution

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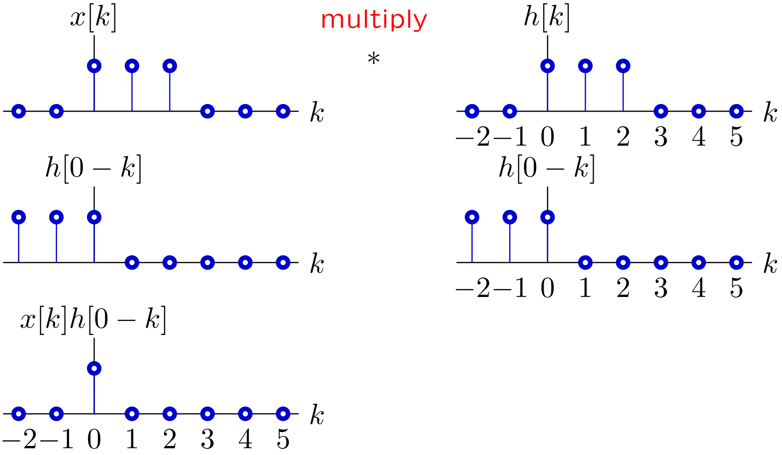
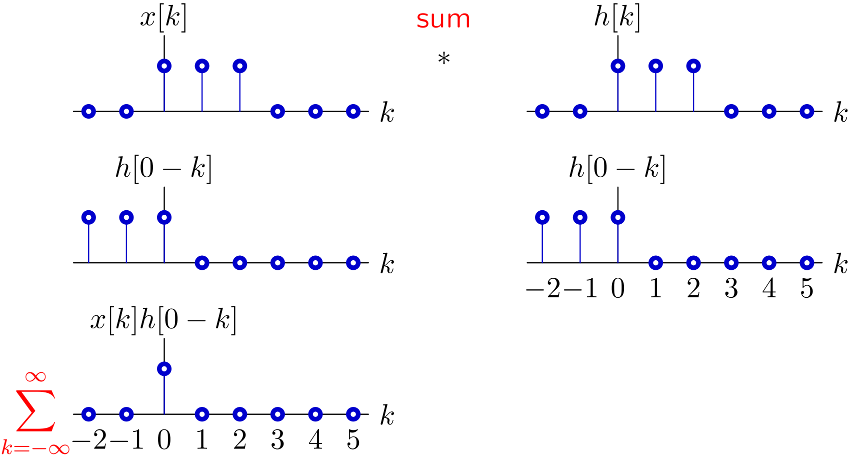


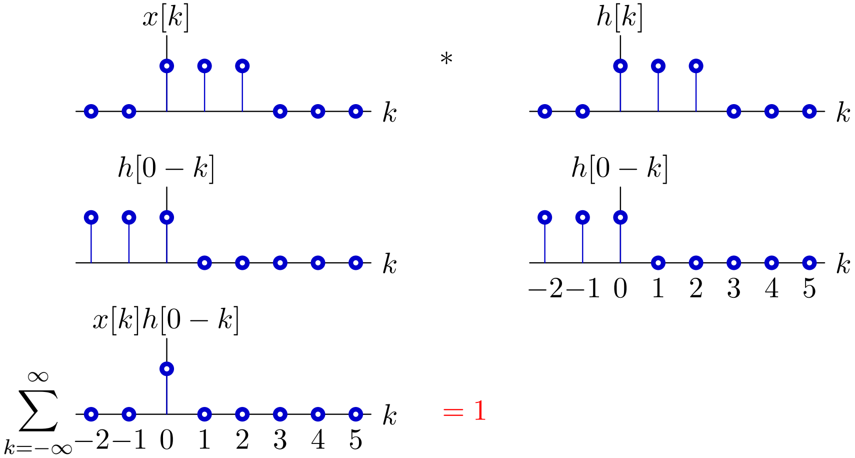
Structure of Convolution

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k]$$



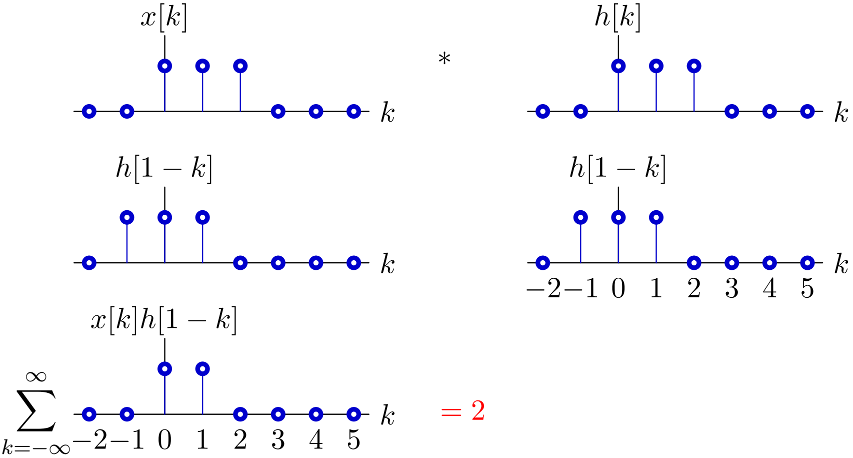
Structure of Convolution

$$y[{\color{red}0}] = \sum_{k=-\infty}^{\infty} x[k]h[0-k]$$



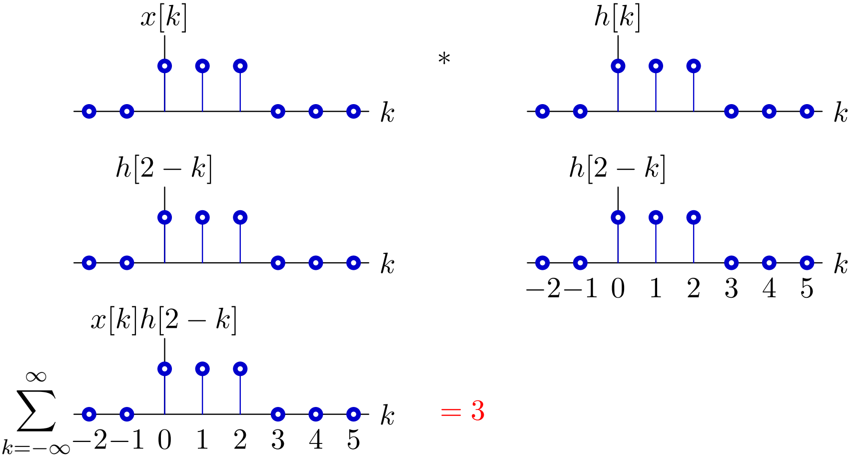
Structure of Convolution

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k]$$



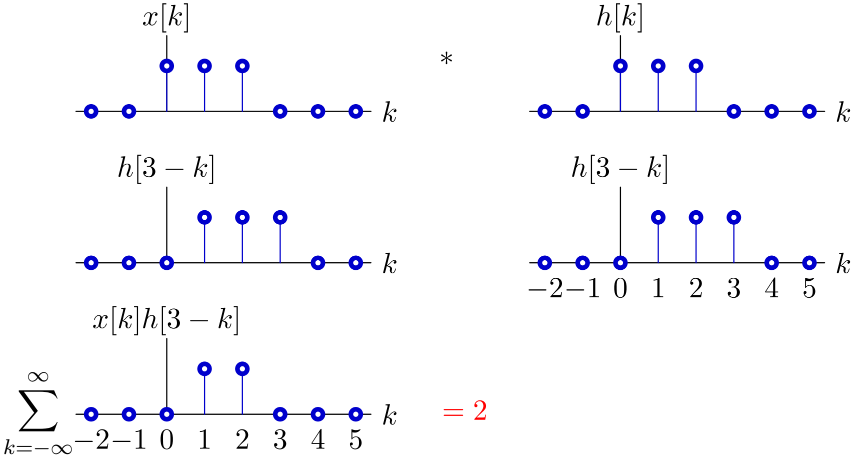
Structure of Convolution

$$y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k]$$



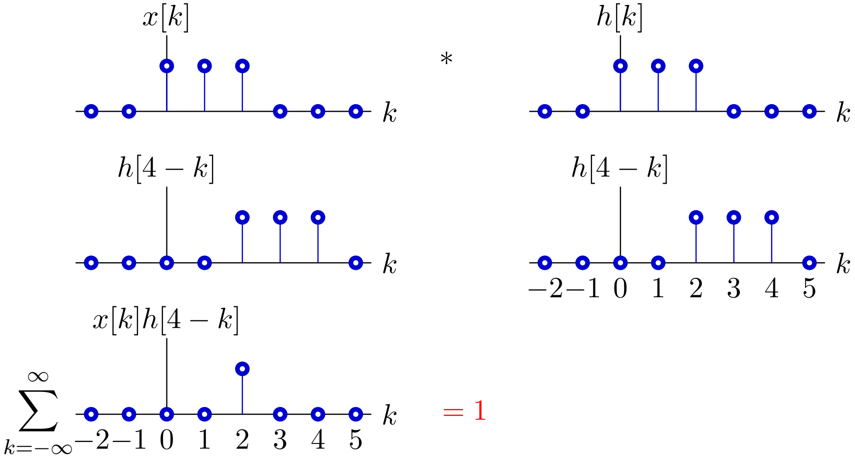
Structure of Convolution

$$y[{\color{red}3}] = \sum_{k=-\infty}^{\infty} x[k]h[3-k]$$



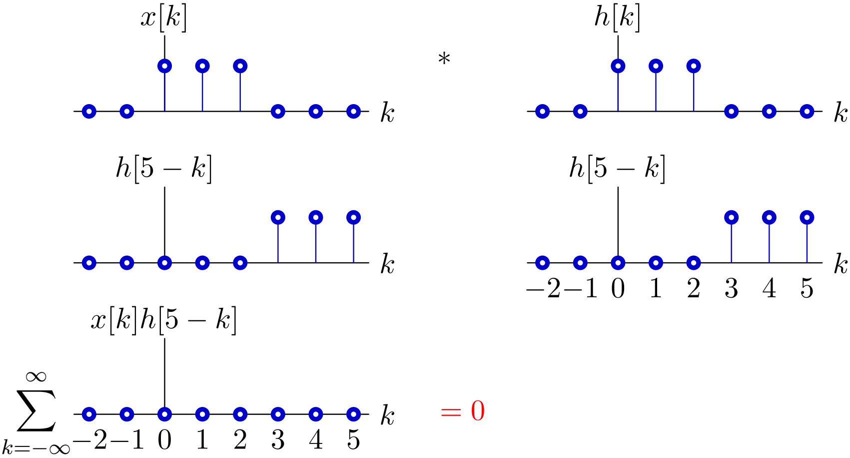
Structure of Convolution

$$y[4] = \sum_{k=-\infty}^{\infty} x[k]h[4-k]$$



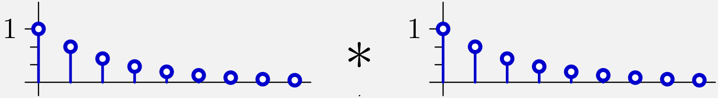
Structure of Convolution

$$y[5] = \sum_{k=-\infty}^{\infty} x[k]h[5-k]$$

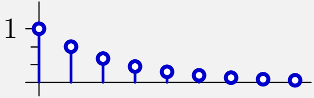


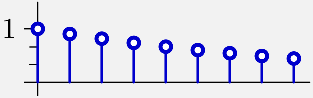
$= 0$

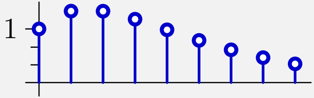
Check Yourself

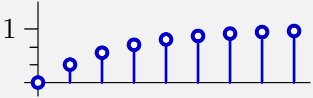


Which plot shows the result of the convolution above?

1. 

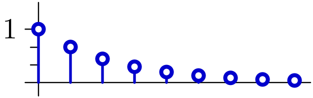
2. 

3. 

4. 

5. none of the above

Check Yourself



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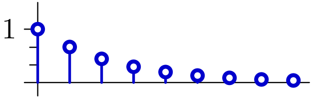


Express mathematically:

$$\left(\left(\frac{2}{3}\right)^n u[n]\right) * \left(\left(\frac{2}{3}\right)^n u[n]\right) = \sum_{k=-\infty}^{\infty} \left(\left(\frac{2}{3}\right)^k u[k]\right) \times \left(\left(\frac{2}{3}\right)^{n-k} u[n-k]\right)$$

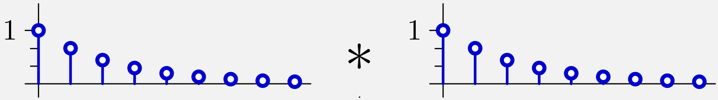
$$= \sum_{k=0}^{n} \left(\frac{2}{3}\right)^k \times \left(\frac{2}{3}\right)^{n-k}$$

$$= \sum_{k=0}^{n} \left(\frac{2}{3}\right)^n = \left(\frac{2}{3}\right)^n \sum_{k=0}^{n} 1$$

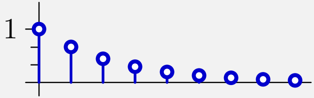
$$= (n+1)\left(\frac{2}{3}\right)^n u[n]$$

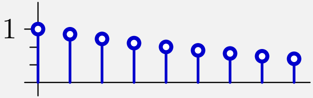
$$= 1,\ \frac{4}{3},\ \frac{4}{3},\ \frac{32}{27},\ \frac{80}{81},\ \ldots$$

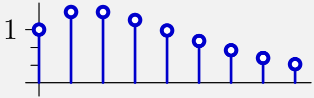
Check Yourself

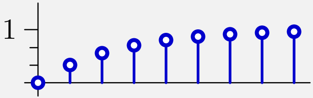


Which plot shows the result of the convolution above? 3

1. 

2. 

3. 

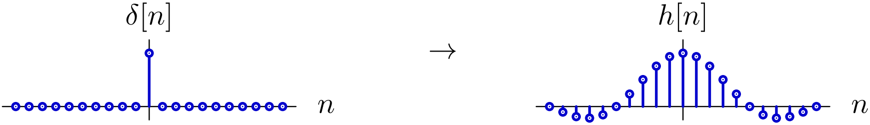
4. 

5. none of the above

Unit-Sample Response

The unit-sample response is a **complete** description of a system.

It can be used to determine the response to **any** other input.



Given $h[n]$ one can compute the response to any arbitrary input signal.

$$y[n] = (x * h)[n] \equiv \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

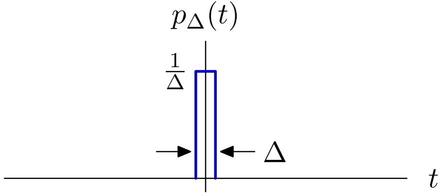
Continuous-Time Systems

Superposition and convolution are of equal importance for CT systems.

Impulse Response

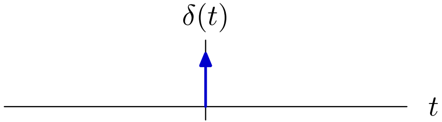
A CT system is completely characterized by its **impulse response**, much as a DT system is completely characterized by its unit-sample response.

We have worked with the impulse (Dirac delta) function $\delta(t)$ previously.
It's defined in a limit as follows.
Let $p_\Delta(t)$ represent a pulse of width Δ and height $\frac{1}{\Delta}$ so that its area is 1.



Then

$$\delta(t) = \lim_{\Delta \to 0} p_\Delta(t)$$

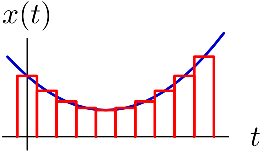


The impulse function can be used to break an arbitrary input $x(t)$ into time-based components, much as $\delta[k]$ is used for discrete-time signals.

Impulse Response

An arbitrary CT signal can be represented by an infinite sum of infinitesimal impulses (which define an integral).

Approximate an arbitrary signal $x(t)$ (blue) as a sum of pulses $p_\Delta(t)$ (red).



$$x_\Delta(t) = \sum_{k=-\infty}^{\infty} x(k\Delta) p_\Delta(t - k\Delta)\Delta$$

and the limit of $x_\Delta(t)$ as $\Delta \to 0$ will approximate $x(t)$.

$$\lim_{\Delta\to 0} x_\Delta(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x(k\Delta) p_\Delta(t - k\Delta)\Delta \to \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau$$

The result in CT is much like the result for DT:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)\, d\tau \qquad\qquad x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta(n - m)$$

Impulse Response

If a system is linear and time-invariant (LTI), its input-output relation is **completely specified** by the system's impulse response $h(t)$.

1. One can always find the impulse response of a system.

$$\delta(t) \longrightarrow \boxed{\text{system}} \longrightarrow h(t)$$

2. Time invariance implies that shifting the input simply shifts the output.

$$\delta(t-\tau) \longrightarrow \boxed{\text{system}} \longrightarrow h(t-\tau)$$

3. Homogeneity implies that scaling the input simply scales the output.

$$x(\tau)\delta(t-\tau) \longrightarrow \boxed{\text{system}} \longrightarrow x(\tau)h(t-\tau)$$

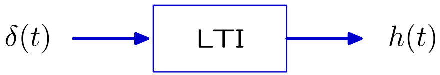
4. Additivity implies that the response to a sum is the sum of responses.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \longrightarrow \boxed{\text{system}} \longrightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \equiv (x*h)(t)$$

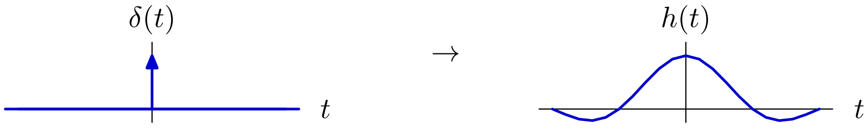
The output of an LTI system can **always** be found by convolving: $(x*h)(t)$.

Impulse Response

The impulse response is a **complete** description of a system.



It can be used to determine the response to **any** other input.



Given $h(t)$ one can compute the response to any arbitrary input signal.

$$y(t) = (x * h)(t) \equiv \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

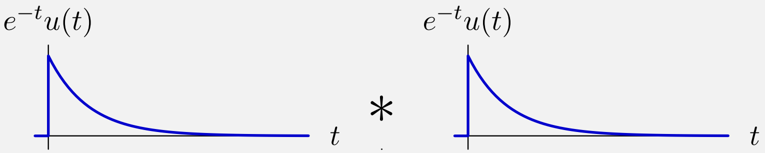
Comparison of CT and DT Convolution

Convolution of CT signals is analogous to convolution of DT signals.

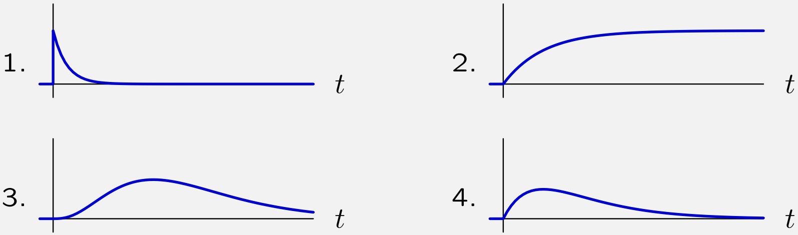
DT: $$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

CT: $$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Check Yourself



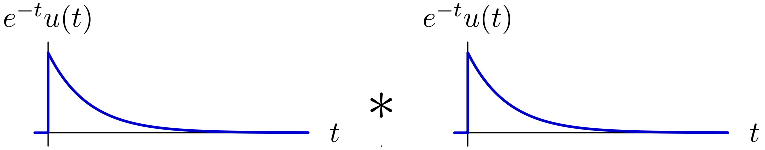
Which plot shows the result of the convolution above?



5. none of the above

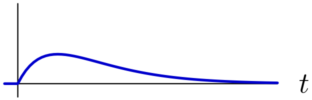
Check Yourself

Which plot shows the result of the following convolution?

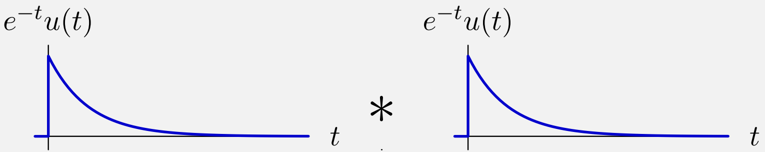


$$\left(e^{-t}u(t)\right) * \left(e^{-t}u(t)\right) = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-(t-\tau)}u(t-\tau)d\tau$$

$$= \int_0^t e^{-\tau}e^{-(t-\tau)}d\tau = e^{-t}\int_0^t d\tau = te^{-t}u(t)$$

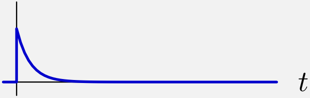


Check Yourself

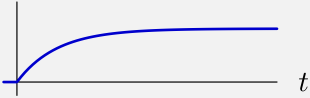


Which plot shows the result of the convolution above? 4

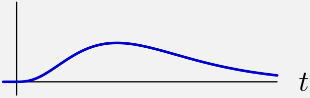
1.



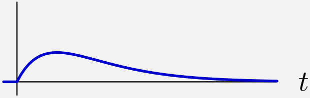
2.



3.



4.



5. none of the above

Properties of Convolution

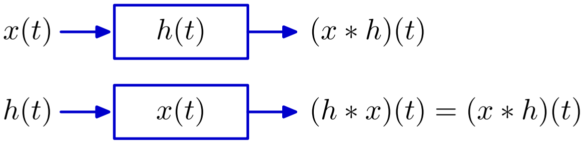
Commutivity:

$$(x * y)(t) = (y * x)(t)$$

$$(x * y)(t) \equiv \int_{-\infty}^{\infty} x(t-\tau)y(\tau)\,d\tau$$

let $\lambda = t - \tau$

$$\begin{aligned}(x * y)(t) &= \int_{\infty}^{-\infty} x(\lambda)y(t-\lambda)(-d\lambda)\\ &= \int_{-\infty}^{\infty} x(\lambda)y(t-\lambda)\,d\lambda\\ &= (y * x)(t)\end{aligned}$$



Properties of Convolution

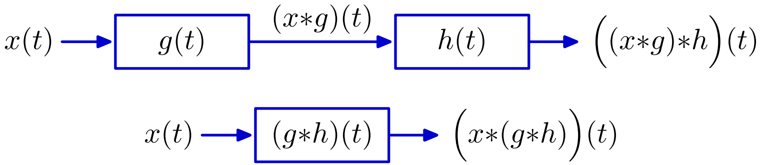
Associativity.

$$\Big((x*y)*z\Big)(t) = \Big(x*(y*z)\Big)(t)$$

$$\Big((x*y)*z\Big)(t) \equiv \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(t-\lambda-\tau)y(\tau)\,d\tau\right)z(\lambda)\,d\lambda$$

let $\mu = \lambda + \tau$

$$\begin{aligned}\Big((x*y)*z\Big)(t) &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(t-\mu)y(\mu-\lambda)\,d\mu\right)z(\lambda)\,d\lambda \\ &= \int_{-\infty}^{\infty} x(t-\mu)\left(\int_{-\infty}^{\infty} y(\mu-\lambda)z(\lambda)\,d\lambda\right)\,d\mu \\ &= \Big(x*(y*z)\Big)\end{aligned}$$



Properties of Convolution

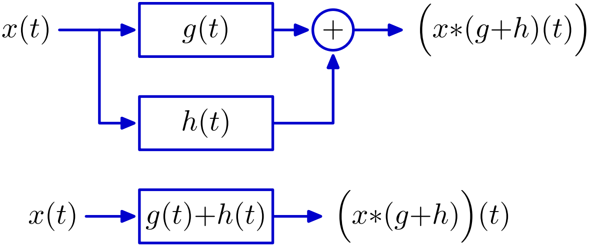
Distributivity over addition.

$$\Big(x * (g + h)\Big)(t) = (x * g)(t) + (x * h)(t)$$

$$\Big(x * (g + h)\Big) = \int_{-\infty}^{\infty} x(t-\tau)\Big(g(\tau) + h(\tau)\Big)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(t-\tau)g(\tau)d\tau + \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

$$= (x * g)(t) + (x * h)(t)$$



Convolution

Convolution is an important **computational tool.**

Example: characterizing LTI systems

- Determine the unit-sample response $h(t)$.
- Calculate the output for an arbitrary input using convolution:

$$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau$$

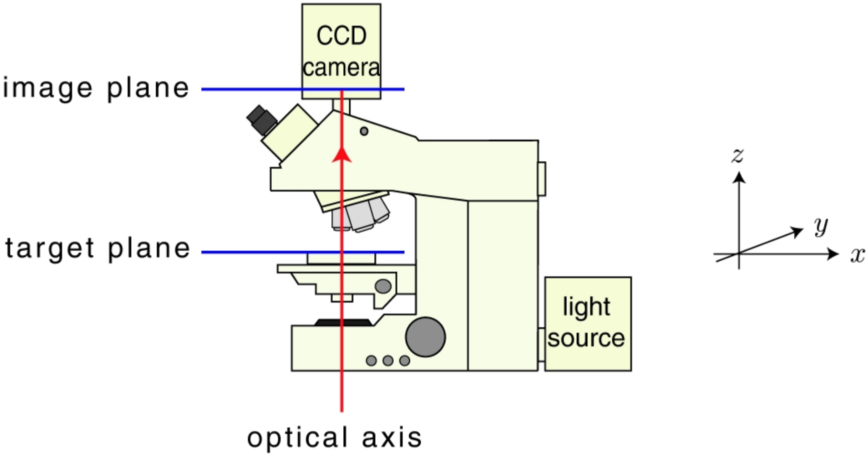
Applications of Convolution

Convolution is an important **conceptual tool:** it provides an important new way to **think** about the behaviors of systems.

Example systems: microscopes and telescopes.

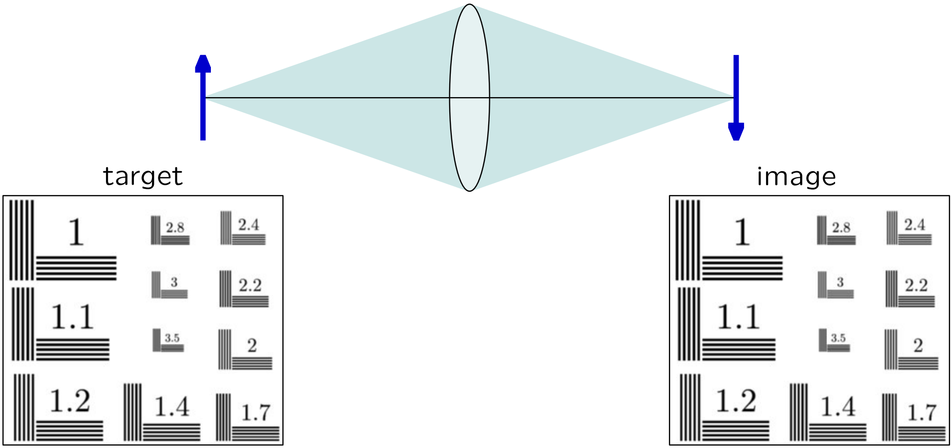
Microscope

Images from even the best microscopes are blurred.



Microscope

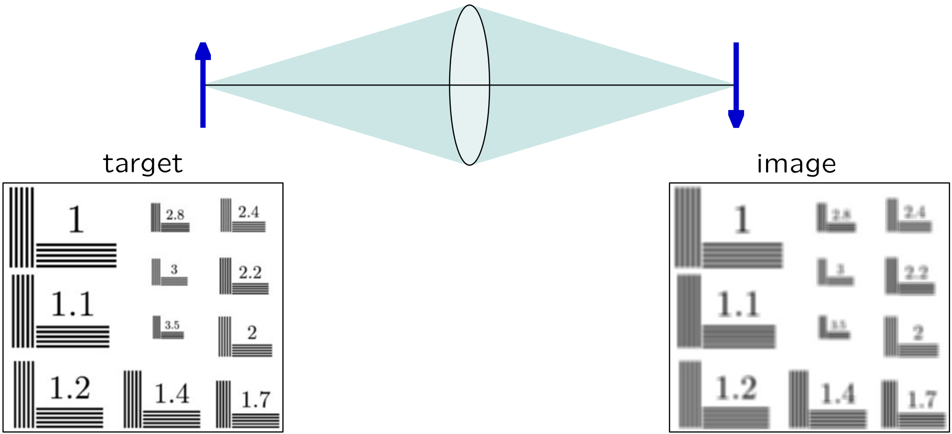
A perfect lens transforms a spherical wave of light from the target into a spherical wave that converges to the image.



Blurring is inversely related to the diameter of the lens.

Microscope

A perfect lens transforms a spherical wave of light from the target into a spherical wave that converges to the image.



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Microscope

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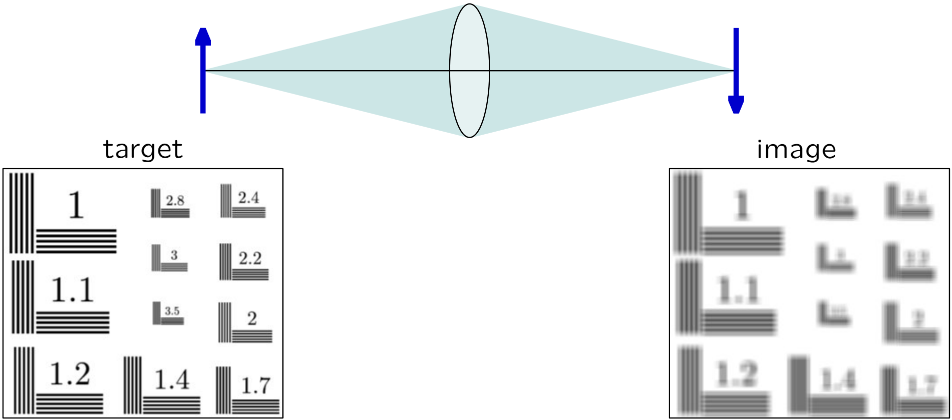
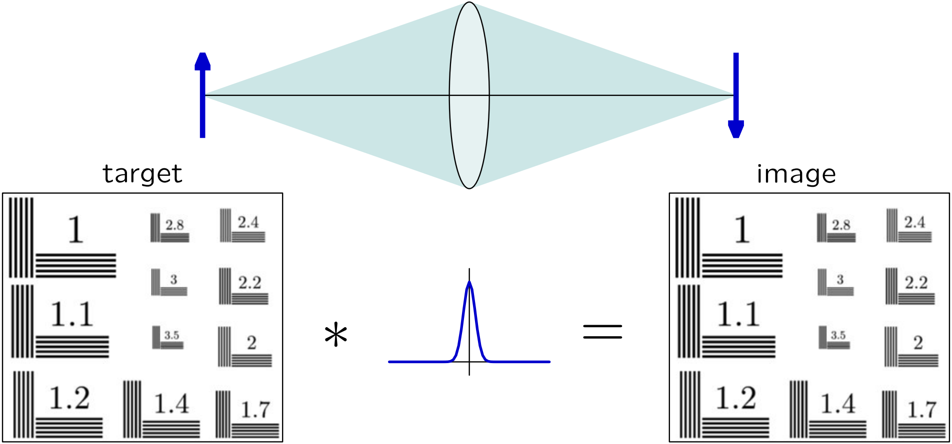


Blurring is inversely related to the diameter of the lens.

Microscope

Blurring can be represented by convolving the image with the optical “point-spread-function” (3D impulse response).



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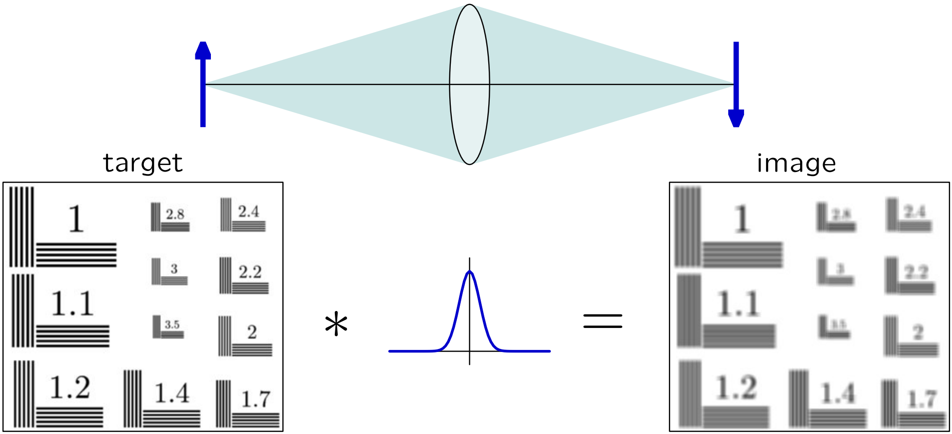

Blurring is inversely related to the diameter of the lens.

Microscope

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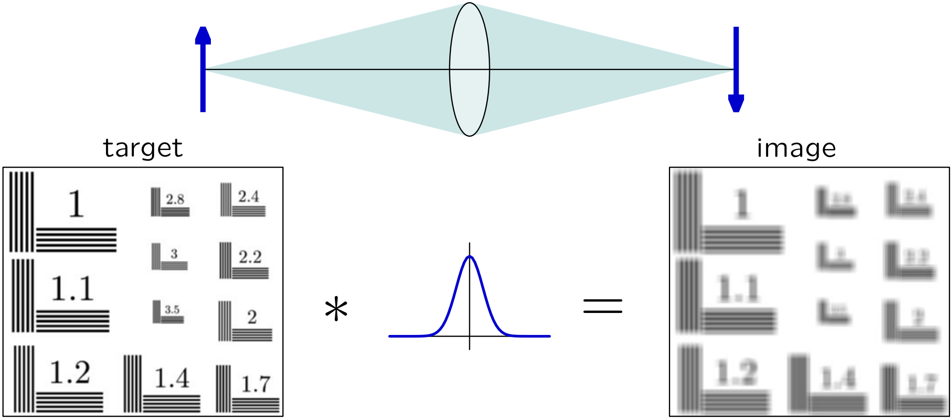


Blurring is inversely related to the diameter of the lens.

Hubble Space Telescope

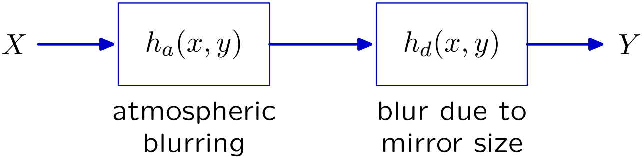
Hubble Space Telescope (1990-)

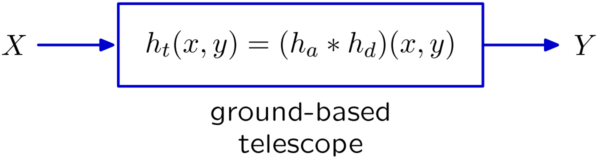
http://hubblesite.org

Hubble Space Telescope

Why build a space telescope?

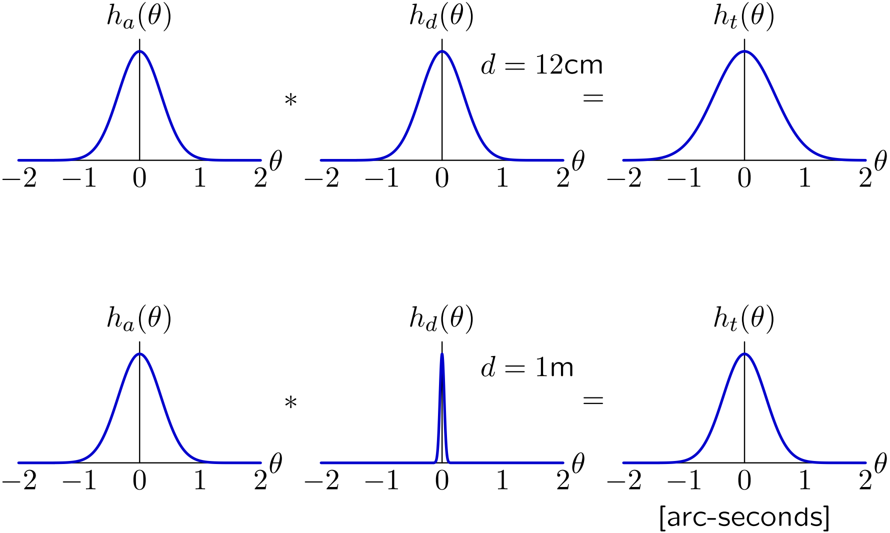
Telescope images are blurred by the telescope lenses AND by atmospheric turbulence.





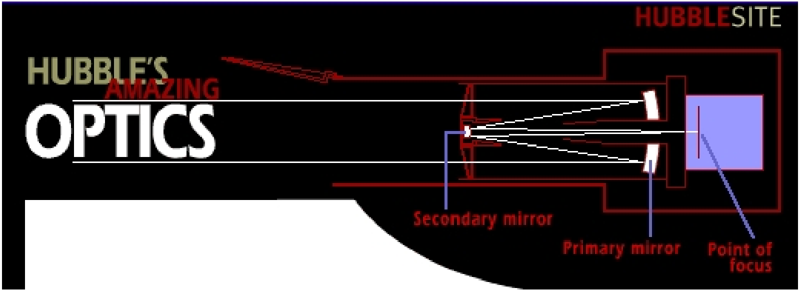
Hubble Space Telescope

Telescope blur can be respresented by the convolution of blur due to atmospheric turbulence and blur due to mirror size.



Hubble Space Telescope

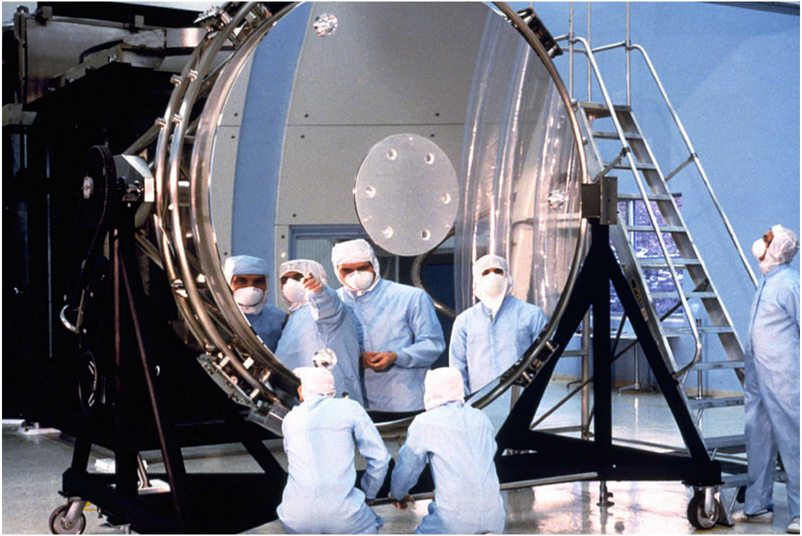
The main optical components of the Hubble Space Telescope are two mirrors.



http://hubblesite.org

Hubble Space Telescope

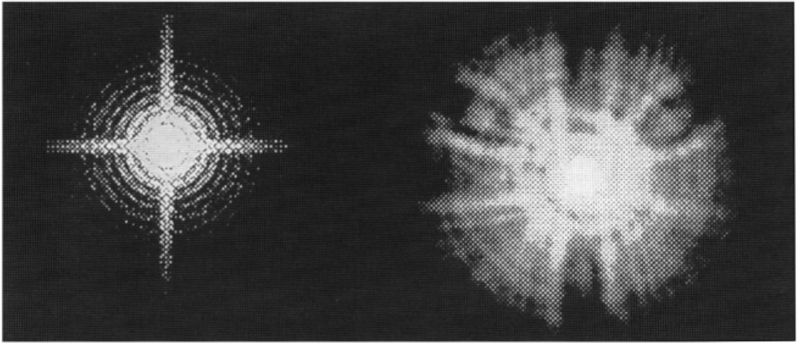
The diameter of the primary mirror is 2.4 meters.



http://hubblesite.org

Hubble Space Telescope

Hubble's first pictures of distant stars (May 20, 1990) were more blurred than expected.



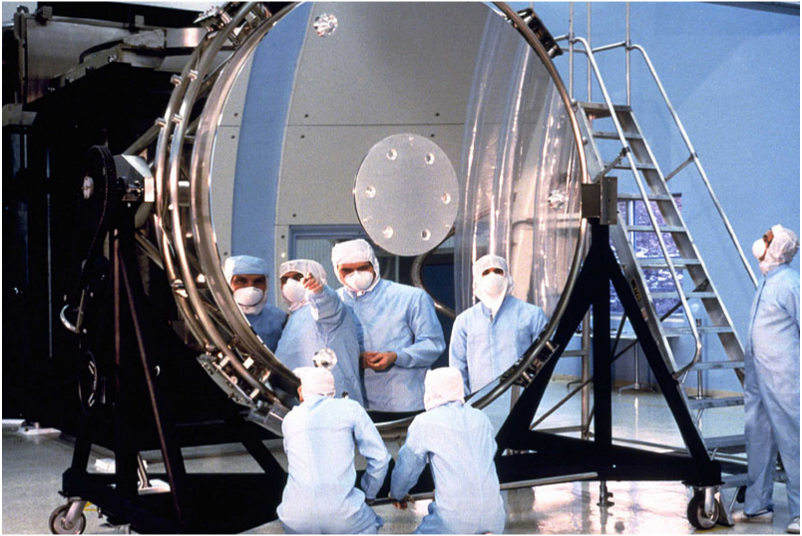
expected point-spread function

early Hubble image of distant star

http://hubblesite.org

Hubble Space Telescope

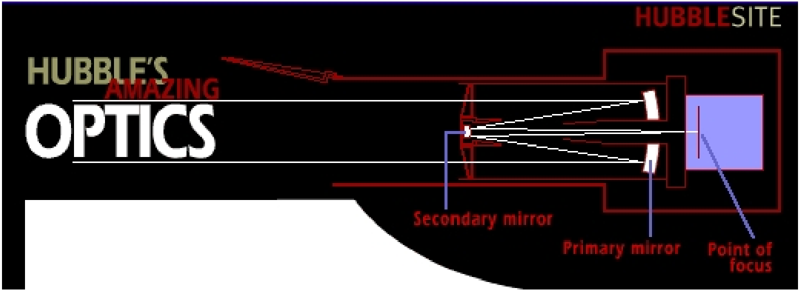
The parabolic mirror was ground 2.2 μm too flat!

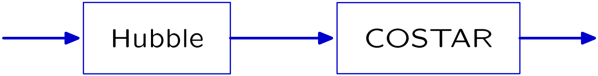


http://hubblesite.org

Hubble Space Telescope

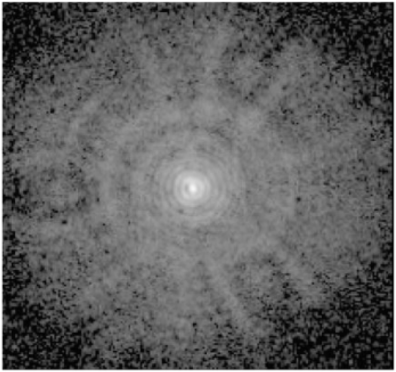
Corrective Optics Space Telescope Axial Replacement (COSTAR): eye-glasses for Hubble!

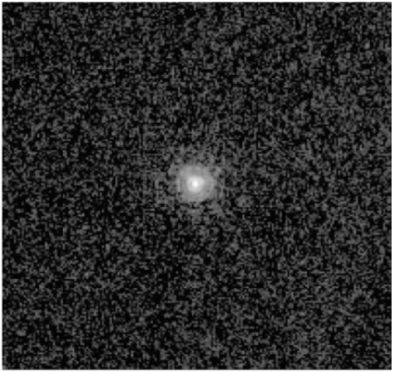




Hubble Space Telescope

Hubble images before and after COSTAR.



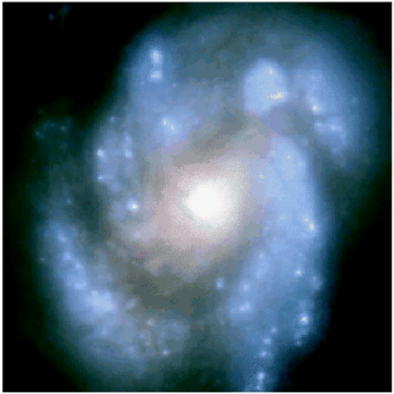


before after

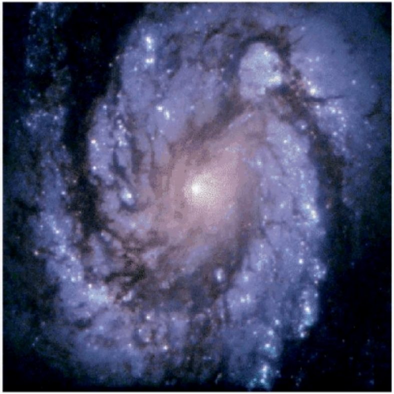
http://hubblesite.org

Hubble Space Telescope

Hubble images before and after COSTAR.



before

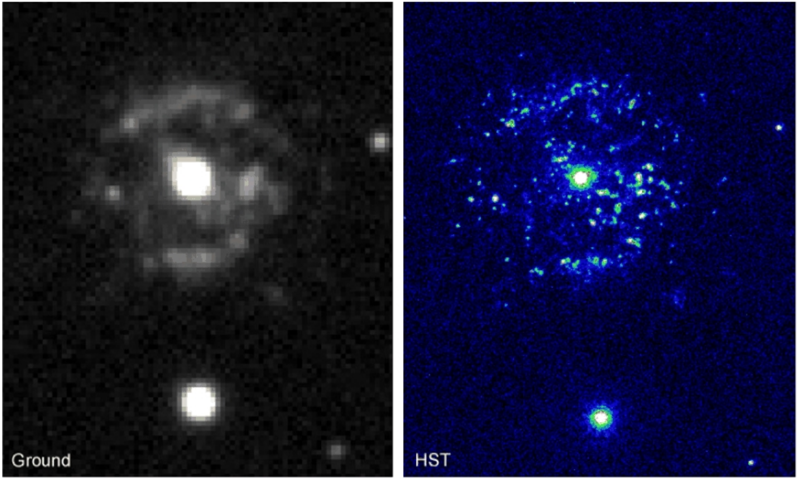


after

http://hubblesite.org

Hubble Space Telescope

Images from ground-based telescope and Hubble.



http://hubblesite.org

Impulse Response: Summary

The impulse response is a complete description of a CT LTI system.

One can find the response to an arbitrary input signal by convolving the input signal with the impulse response.

The impulse response is an especially useful description of some types of systems, e.g., optical systems, where blurring is an important figure of merit.