6.003: Signal Processing

Discrete-Time Fourier Transform

• Definition
• Examples
• Properties
• Relations between Fourier series and transforms (DT and CT)

Announcements:
• No PSet 4 – practice quiz questions have been posted instead.
• Office Hours today (7-9pm) and Sunday (4-6pm)
  – practice quiz problems and general questions
• Solutions to practice quiz questions will be posted Sunday at 6pm
• Quiz 1: Tuesday, March 3, 2-4pm in 32-141.
  – Coverage up to and including all of week 4.
  – Closed book except for one page of notes (8.5”x11” both sides).
  – No electronic devices. (No headphones, cellphones, calculators, ...)

February 27, 2020
From Periodic to Aperiodic

Last time: representing arbitrary (aperiodic) CT signals as sums of sinusoidal components using the continuous-time Fourier transform.

Today: generalize the Fourier Transform idea to discrete-time signals.
Fourier Representations of Aperiodic Signals

How can we represent an aperiodic signal as a sum of sinusoids?

Strategy: make a periodic version of $x[n]$ by summing shifted copies:

$$x_p[n] = \sum_{m=-\infty}^{\infty} x[n - mN]$$

Since $x_p[n]$ is periodic, it has a Fourier series (which depends on $N$).

Find Fourier series coefficients $X_p[k]$ and take the limit of $X_p[k]$ as $N \to \infty$.

As $N \to \infty$, $x_p[n] \to x[n]$, and Fourier series will approach Fourier transform.
Fourier Representations of Aperiodic Signals

Example.

Strategy: make a periodic version of $x[n]$ by summing shifted copies:

$$x_p[n] = \sum_{m=-\infty}^{\infty} x[n - mN]$$

Calculate the Fourier series coefficients $X_p[k]$:

$$X_p[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x_p[n] e^{-j \frac{2\pi k n}{N}} = \frac{1}{N} + \frac{2}{N} \cos \frac{2\pi k}{N} + \frac{2}{N} \cos \frac{4\pi k}{N}$$
Fourier Representations of Aperiodic Signals

Calculate the Fourier series coefficients $X_p[k]$:

$$X_p[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x_p[n] e^{-j \frac{2\pi}{N} k n} = \frac{1}{N} + \frac{2}{N} \cos \frac{2\pi k}{N} + \frac{2}{N} \cos \frac{4\pi k}{N}$$

Plot the resulting Fourier coefficients for $N=8$.

What happens if you double the period $N$?
Fourier Representations of Aperiodic Signals

Calculate the Fourier series coefficients $X_p[k]$:

$$X_p[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x_p[n] e^{-j\frac{2\pi}{N} kn} = \frac{1}{N} + \frac{2}{N} \cos \frac{2\pi k}{N} + \frac{2}{N} \cos \frac{4\pi k}{N}$$

Plot the resulting Fourier coefficients for $N=8$.

What happens if you double the period $N$?
- The amplitude will shrink by a factor of 2.
- There will be twice as many samples per period of the cosine functions.

The red samples are at new intermediate frequencies.
Define a new function $X(\Omega) = NX_p[k]$ where $\Omega = \frac{2\pi k}{N}$.

\[
NX_p[k] = 1 + 2 \cos \frac{2\pi k}{N} + 2 \cos \frac{4\pi k}{N} = 1 + 2 \cos(\Omega) + 2 \cos(2\Omega) = X(\Omega)
\]

The discrete function $NX_p[k]$ is a sampled version of the function $X(\Omega)$. 
Fourier Representations of Aperiodic Signals

We can reconstruct $x[n]$ from $X(\Omega)$ using Riemann sums.

$$x_p[n] = \sum_{k=\langle N \rangle} X_p[k] e^{j \frac{2\pi}{N} kn} = \frac{1}{2\pi} \sum_{k=\langle N \rangle} N X_p[k] e^{j \frac{2\pi}{N} kn} \left( \frac{2\pi}{N} \right)$$

$$x[n] = \lim_{N \to \infty} x_p[n] = \lim_{N \to \infty} \frac{1}{2\pi} \sum_{k=\langle N \rangle} N X_p[k] e^{j \frac{2\pi}{N} kn} \left( \frac{2\pi}{N} \right) = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j \Omega n} d\Omega$$

$$N X_p[k] = X(\Omega)$$

Fourier Transform relation: $x[n] \overset{\text{FT}}{\iff} X(\Omega)$
Fourier Series and Fourier Transform

Fourier series and transforms are similar: both represent signals by their frequency content.

Discrete-Time Fourier Series

\[ X[k] = X[k+N] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j k \Omega_0 n} \]  
analysis equation

\[ x[n] = x[n+N] = \sum_{k=\langle N \rangle} X[k] e^{j k \Omega_0 n} \]  
synthesis equation

where \( \Omega_0 = \frac{2\pi}{N} \)

Discrete-Time Fourier Transform

\[ X(\Omega) = X(\Omega+2\pi) = \sum_{n=-\infty}^{\infty} x[n] e^{-j \Omega n} \]  
analysis equation

\[ x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j \Omega n} \, d\Omega \]  
synthesis equation
Fourier Series and Fourier Transform

All of the information in a periodic signal is contained in one period. The information in an aperiodic signal is spread across time.

Discrete-Time Fourier Series

\[ X[k] = X[k+N] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\Omega_0 n} \]

analysis equation

\[ x[n] = x[n+N] = \sum_{k=\langle N \rangle} X[k] e^{jk\Omega_0 n} \]

synthesis equation

where \( \Omega_0 = \frac{2\pi}{N} \)

Discrete-Time Fourier Transform

\[ X(\Omega) = X(\Omega+2\pi) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \]

analysis equation

\[ x[n] = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} X(\Omega) e^{j\Omega n} \, d\Omega \]

synthesis equation
Fourier Series and Fourier Transform

Periodic signals can be synthesized from a discrete set of $k$ harmonics. Aperiodic signals generally require a continuous set of frequencies $\Omega$.

**Discrete-Time Fourier Series**

$X[k] = X[k+N] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n]e^{-jk\Omega_0 n}$  \hspace{1cm} \text{analysis equation}

$x[n] = x[n+N] = \sum_{k=\langle N \rangle} X[k]e^{jk\Omega_0 n}$  \hspace{1cm} \text{synthesis equation}

where $\Omega_0 = \frac{2\pi}{N}$

**Discrete-Time Fourier Transform**

$X(\Omega) = X(\Omega+2\pi) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$  \hspace{1cm} \text{analysis equation}

$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n} \, d\Omega$  \hspace{1cm} \text{synthesis equation}
Fourier Series and Fourier Transform

Harmonic frequencies $k\Omega_o$ are samples of continuous frequency $\Omega$.

### Discrete-Time Fourier Series

- **Analysis Equation**
  \[
  X[k] = X[k+N] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\Omega_on}
  \]

- **Synthesis Equation**
  \[
  x[n] = x[n+N] = \sum_{k=\langle N \rangle} X[k] e^{jk\Omega_on}
  \]

### Discrete-Time Fourier Transform

- **Analysis Equation**
  \[
  X(\Omega) = X(\Omega+2\pi) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}
  \]

- **Synthesis Equation**
  \[
  x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega
  \]
CT and DT Fourier Transforms

DT frequencies alias because adding $2\pi$ to $\Omega$ does not change $e^{-j\Omega n}$.

Continuous-Time Fourier Transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$  \hspace{1cm} \text{analysis equation}

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$  \hspace{1cm} \text{synthesis equation}

Discrete-Time Fourier Transform

$$X(\Omega) = X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$  \hspace{1cm} \text{analysis equation}

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega$$  \hspace{1cm} \text{synthesis equation}
CT and DT Fourier Transforms

DT frequencies alias because adding $2\pi$ to $\Omega$ does not change $e^{-j\Omega n}$. Because of aliasing, we need only integrate $d\Omega$ over a $2\pi$ interval.

Continuous-Time Fourier Transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

analysis equation

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$
synthesis equation

Discrete-Time Fourier Transform

$$X(\Omega) = X(\Omega+2\pi) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

analysis equation

$$x[n] = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} X(\Omega)e^{j\Omega n}d\Omega$$
synthesis equation
Examples of Fourier Transforms

Find the Fourier Transform (FT) of a rectangular pulse of width $2S+1$:

$$p_S[n] = \begin{cases} 1 & -S \leq N \leq S \\ 0 & \text{otherwise} \end{cases}$$

We would like to reduce this expression to a closed form to better identify trends across $S$. 

$$P_S(\Omega) = \sum_{n=-\infty}^{\infty} p_S[n]e^{-j\Omega n} = \sum_{n=-S}^{S} e^{-j\Omega n}$$

$$= e^{j\Omega S} + e^{j\Omega(S-1)} + \cdots + 1 + \cdots + e^{-j\Omega(S-1)} + e^{-j\Omega S}$$

$$= 1 + 2\cos(\Omega) + 2\cos(2\Omega) + \cdots + 2\cos(S\Omega)$$
Working with Sums

Closed form sums of exponential sequences.

\[ A = \sum_{n=0}^{N-1} \alpha^n \]

If the series has finite length (here \( N \) terms), it will converge for finite \( \alpha \).

\[
\begin{align*}
A &= 1 + \alpha + \alpha^2 + \cdots + \alpha^{N-1} \\
\alpha A &= \alpha + \alpha^2 + \cdots + \alpha^{N-1} + \alpha^N \\
A - \alpha A &= 1 - \alpha^N
\end{align*}
\]

\[
A = \begin{cases} 
\frac{1 - \alpha^N}{1 - \alpha} & \text{if } \alpha \neq 1 \\
N & \text{if } \alpha = 1
\end{cases}
\]

If the series has infinite length, it will converge if \( |\alpha| < 1 \).

\[
\sum_{n=0}^{\infty} \alpha^n = \lim_{N \to \infty} \sum_{n=0}^{N-1} \alpha^n = \lim_{N \to \infty} \frac{1 - \alpha^N}{1 - \alpha} = \frac{1}{1 - \alpha} \quad \text{if } |\alpha| < 1
\]
Examples of Fourier Transforms

Find the Fourier Transform (FT) of a rectangular pulse of width $2S+1$:

$$p_S[n] = \begin{cases} 
1 & -S \leq N \leq S \\
0 & \text{otherwise} 
\end{cases}$$

$$P_S(\Omega) = \sum_{n=-S}^{S} e^{-j\Omega n} = \left(e^{j\Omega S}\right) \sum_{n=0}^{2S} e^{-j\Omega n} = \left(\frac{e^{j\Omega(S+\frac{1}{2})}}{e^{j\Omega/2}}\right) \frac{1 - e^{-j\Omega(2S+1)}}{1 - e^{-j\Omega}}$$

$$= \left(\frac{e^{j\Omega(S+\frac{1}{2})} - e^{-j\Omega(S+\frac{1}{2})}}{e^{j\Omega/2} - e^{-j\Omega/2}}\right) = \frac{\sin \left(\Omega(S+\frac{1}{2})\right)}{\sin \left(\Omega/2\right)}$$
Examples of Fourier Transforms

Compare Fourier transforms of pulses with different widths.

As the function widens in $n$ the Fourier transform narrows in $\Omega$. 
Moments

Similar to CT, the value of $X(\Omega)$ at $\Omega = 0$ is the sum of $x[n]$ over time $t$.

$$X(0) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x[n]$$
Moments

The value of $x[0]$ is $\frac{1}{2\pi}$ times the integral of $X(\Omega)$ over $\Omega = [-\pi, \pi]$.

$$x[0] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega = \frac{1}{2\pi} \int_{2\pi} X(\Omega) d\Omega$$

Notice that the form of this relation is similar to the one for CT at the end of last lecture (and repeated in the next slide).
Moments

The value of $x(0)$ is the integral of $X(\omega)$ divided by $2\pi$.

$$x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) d\omega$$
Stretching Time

Stretching time compresses frequency and increases amplitude (preserving area).

\[ X_1(\omega) = \frac{2 \sin \omega}{\omega} \]

Very similar in CT and DT.
Compressing Time to the Limit

Alternatively, we could compress time while keeping area $= 1$.

\[ x(t) = \begin{cases} 0 & \text{for } -\frac{1}{2} < t < 0 \\ 1 & \text{for } 0 < t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \]

\[ X(\omega) = \frac{\sin \omega/2}{\omega/2} \]

In the limit, the pulse has zero width but area 1!

We represent this limit with the delta function: $\delta(t)$. 
Math With Impulses

Although physically unrealizable, the impulse (a.k.a. Dirac delta) function is useful as a mathematically tractable approximation to a very brief signal.

Example 1: Find the Fourier transform of a unit impulse function.

\[ X(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t} \, dt \]

Since \( \delta(t) \) is zero except near \( t=0 \), only values of \( e^{-j\omega t} \) near \( t=0 \) are important. Because \( e^{-j\omega t} \) is a smooth function of \( t \), \( e^{-j\omega t} \) can be replaced by \( e^{-j\omega 0} \):

\[ X(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega 0} \, dt = 1 \]

This matches our previous result which was based explicitly on a limit. Here the limit is implicit.
Although physically unrealizable, the impulse function is extremely useful as a mathematically tractable approximation to a very brief signal.

Example 2: Find the function whose Fourier transform is a unit impulse.

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega)e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega)e^{j0t} d\omega = \frac{1}{2\pi} \]

\[ 1 \quad \overset{\text{CTFT}}{\Longleftrightarrow} \quad 2\pi \delta(\omega) \]

Notice the similarity to the previous result:

\[ \delta(t) \quad \overset{\text{CTFT}}{\Longleftrightarrow} \quad 1 \]

These relations are duals of each other.

- A constant in time consists of a single frequency at \( \omega = 0 \).
- An impulse in time contains components at all frequencies.
Math With Impulses

Delta functions are similarly useful in discrete-time transforms.

Let

\[ X(\Omega) = \sum_{m=-\infty}^{\infty} \delta(\Omega - 2\pi m) \]

where the sum results because DT Fourier Transforms are periodic in \(2\pi\).

Then

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n} d\Omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(\Omega) e^{j\Omega n} d\Omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(\Omega) d\Omega = \frac{1}{2\pi} \]

Thus if \(x[n] = 1\) for all \(n\), the transform is a delta function in frequency.

\[ 1 \overset{\text{DTFT}}{\iff} \sum_{m=-\infty}^{\infty} 2\pi \delta(\Omega - 2\pi m) \]

We previously showed a similar result for CT:

\[ 1 \overset{\text{CTFT}}{\iff} 2\pi \delta(\omega) \]
Math With Impulses

Using delta functions to calculate the transforms of complex exponentials. Let

\[ X(\omega) = \delta(\omega - \omega_0) \]

then

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega \]

The delta function is zero except when \( \omega = \omega_0 \). Therefore we can substitute \( e^{j\omega_0 t} \) for \( e^{j\omega t} \) without changing the result of the integration.

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega_0 t} d\omega = \frac{1}{2\pi} e^{j\omega_0 t} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) d\omega = \frac{1}{2\pi} e^{j\omega_0 t} \]

Thus the transform of a complex exponential is a delta at that frequency.

\[ e^{j\omega_0 t} \overset{\text{CTFT}}{\leftrightarrow} 2\pi \delta(\omega - \omega_0) \]

The analogous expression for DT holds using analogous reasoning.

\[ e^{j\Omega_0 n} \overset{\text{DTFT}}{\leftrightarrow} \sum_{m=-\infty}^{\infty} 2\pi \delta(\Omega - \Omega_0 - 2\pi m) \]
Relations Between Fourier Series and Fourier Transforms

Continuous Time:

\[ e^{j\frac{2\pi k}{T}t} \quad \text{CTFT} \quad 2\pi \delta \left( \omega - \frac{2\pi k}{T} \right) \]

\[ x(t) = x(t+T) \quad \text{CTFS} \quad X[k] \]

\[ x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi k}{T}t} \quad \text{CTFT} \quad \sum_{k=-\infty}^{\infty} 2\pi X[k] \delta \left( \omega - \frac{2\pi k}{T} \right) \]

Discrete Time:

\[ e^{j\frac{2\pi k}{N}n} \quad \text{DTFT} \quad \sum_{m=-\infty}^{\infty} 2\pi \delta \left( \Omega - \frac{2\pi k}{N} - 2\pi m \right) \]

\[ x[n] = x[n+N] \quad \text{DTFS} \quad X[k] \]

\[ x[n] = x[n+N] = \sum_{k=\langle N \rangle} X[k] e^{j\frac{2\pi k}{N}n} \quad \text{DTFT} \quad \sum_{k=\langle N \rangle} \sum_{m=-\infty}^{\infty} 2\pi X[k] \delta \left( \Omega - \frac{2\pi k}{N} - 2\pi m \right) \]
Relation between Fourier Transform and Fourier Series

Each term in the Fourier series is replaced by an impulse in $\omega$.

\[ x(t) = \sum_{k=-\infty}^{\infty} x_p(t - kT) \]

\[ X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi X[k] \delta(\omega - k \frac{2\pi}{T}) \]
Summary

Discrete-Time Fourier Transform

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• Examples
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