6.003: Signal Processing

Discrete-Time Fourier Transform

- Definition
- Examples
- Properties
- Relations between Fourier series and transforms (DT and CT)

Announcements:
- No PSet 4 – practice quiz questions have been posted instead.
- Office Hours today (7-9pm) and Sunday (4-6pm)
  - practice quiz problems and general questions
  - Closed book except for one page of notes (8.5”x11” both sides).
  - No electronic devices. (No headphones, cellphones, calculators, …)

From Periodic to Aperiodic

Last time: representing arbitrary (aperiodic) CT signals as sums of sinusoidal components using the continuous-time Fourier transform.

Today: generalize the Fourier Transform idea to discrete-time signals.

Fourier Representations of Aperiodic Signals

How can we represent an aperiodic signal as a sum of sinusoids?

\[
x[n]
\]

Strategy: make a periodic version of \( x[n] \) by summing shifted copies:

\[
x_p[n] = \sum_{m=-\infty}^{\infty} x[n - mN]
\]

Since \( x_p[n] \) is periodic, it has a Fourier series (which depends on \( N \)).

Find Fourier series coefficients \( X_p[k] \) and take the limit of \( X_p[k] \) as \( N \to \infty \).

As \( N \to \infty \), \( x_p[n] \to x[n] \), and Fourier series will approach Fourier transform.

Fourier Representations of Aperiodic Signals

Calculate the Fourier series coefficients \( X_p[k] \):

\[
X_p[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x_p[n] e^{-j \frac{2 \pi k n}{N}} = \frac{1}{N} + \frac{2}{N} \cos \frac{2 \pi k}{N} + \frac{2}{N} \cos \frac{4 \pi k}{N}
\]

Plot the resulting Fourier coefficients for \( N=8 \).

What happens if you double the period \( N \)?

- The amplitude will shrink by a factor of 2.
- There will be twice as many samples per period of the cosine functions.

The discrete function \( NX_p[k] \) is a sampled version of the function \( X(\Omega) \).

Fourier Representations of Aperiodic Signals

Define a new function \( X(\Omega) = NX_p[k] \) where \( \Omega = 2 \pi k / N \).

\[
NX_p[k] = 1 + 2 \cos \frac{2 \pi k}{N} + 2 \cos \frac{4 \pi k}{N} = 1 + 2 \cos(\Omega) + 2 \cos(2\Omega) = X(\Omega)
\]

- \( N=8 \):
  - \( \Omega = \frac{2 \pi k}{N} \)
- \( N=16 \):
  - \( \Omega = \frac{2 \pi k}{N} \)
- \( N=32 \):
  - \( \Omega = \frac{2 \pi k}{N} \)

The discrete function \( NX_p[k] \) is a sampled version of the function \( X(\Omega) \).
Fourier Representations of Aperiodic Signals

We can reconstruct \( x[n] \) from \( X(\Omega) \) using Riemann sums.

\[
x_p[n] = \sum_{k=0}^{N-1} X_p[k] e^{\frac{j \Omega n}{N}} = \frac{1}{N} \sum_{k=0}^{N-1} X_p[k] e^{\frac{j \Omega n}{N}}
\]

\[
x[n] = \lim_{N \to \infty} x_p[n] = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} N X_p[k] e^{\frac{j \Omega n}{N}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega
\]

| \( N \)| | \( \Omega \)| | \( \Omega = 2\pi k \)|
|---|---|---|
| 8 | \( -2\pi \) | \( -\pi \) |
| 16 | \( -2\pi \) | \( -\pi \) |
| 32 | \( -2\pi \) | \( -\pi \) |

Fourier Transform relation: \( x[n] \xrightarrow{\mathcal{F}} X(\Omega) \)

Discrete-Time Fourier Series

\[
x[k] = X[k+N] = \frac{1}{N} \sum_{n=-N}^{N-1} x[n] e^{-j\Omega kn} \quad \text{analysis equation}
\]

\[
x[n] = x[n+N] = \sum_{k=-N}^{N-1} X[k] e^{j\Omega kn} \quad \text{synthesis equation}
\]

Discrete-Time Fourier Transform

\[
x[k] = X(k+2\pi) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega kn} \quad \text{analysis equation}
\]

\[
x[n] = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} X(\Omega) e^{j\Omega n} d\Omega \quad \text{synthesis equation}
\]

Fourier Series and Fourier Transform

All of the information in a periodic signal is contained in one period. The information in an aperiodic signal is spread across time.

Continuous-Time Fourier Series

\[
x(t) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \text{analysis equation}
\]

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \quad \text{synthesis equation}
\]

Continuous-Time Fourier Transform

\[
x(t) = X(t+2\pi) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \quad \text{analysis equation}
\]

\[
x[n] = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} X(\Omega) e^{j\Omega n} d\Omega \quad \text{synthesis equation}
\]

Fourier Series and Fourier Transform

Both represent signals by their frequency content.

Periodic signals can be synthesized from a discrete set of \( k \) harmonics. Aperiodic signals generally require a continuous set of frequencies \( \Omega \).

CT and DT Fourier Transforms

DT frequencies alias because adding \( 2\pi \) to \( \Omega \) does not change \( e^{-j\Omega n} \).

Continuous-Time Fourier Transform

\[
x(t) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \text{analysis equation}
\]

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \quad \text{synthesis equation}
\]

Continuous-Time Fourier Transform

\[
x(t) = X(t+2\pi) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \quad \text{analysis equation}
\]

\[
x[n] = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} X(\Omega) e^{j\Omega n} d\Omega \quad \text{synthesis equation}
\]
CT and DT Fourier Transforms

DT frequencies alias because adding 2π to Ω does not change e^{-j2πn}. Because of aliasing, we only integrate df over a 2π interval.

Continuous-Time Fourier Transform

\[
X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \quad \text{analysis equation}
\]

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega \quad \text{synthesis equation}
\]

Discrete-Time Fourier Transform

\[
X(\Omega) = X(\Omega+2\pi) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \quad \text{analysis equation}
\]

\[
x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n} d\Omega \quad \text{synthesis equation}
\]

Examples of Fourier Transforms

Find the Fourier Transform (FT) of a rectangular pulse of width 2S+1:

\[
p_S[n] = \begin{cases} 
1 & -S \leq N \leq S \\
0 & \text{otherwise}
\end{cases}
\]

\[
X(\Omega) = \sum_{n=-\infty}^{\infty} p_S[n]e^{-j\Omega n} = \sum_{n=-S}^{S} e^{-j\Omega n} = e^{j\Omega S + e^{j(\Omega S-1)} + \cdots + 1 + \cdots + e^{-j(\Omega S-1)} + e^{-j\Omega S}} = 1 + 2\cos(\Omega) + 2\cos(2\Omega) + \cdots + 2\cos(S\Omega)
\]

We would like to reduce this expression to a closed form to better identify trends across S.

Examples of Fourier Transforms

Find the Fourier Transform (FT) of a rectangular pulse of width 2S+1:

\[
p_S[n] = \begin{cases} 
1 & -S \leq N \leq S \\
0 & \text{otherwise}
\end{cases}
\]

\[
P_S(\Omega) = \sum_{n=-\infty}^{\infty} p_S[n]e^{-j\Omega n} = \sum_{n=-S}^{S} e^{-j\Omega n} = \frac{e^{j\Omega(S+\frac{1}{2})} - e^{-j\Omega(S+\frac{1}{2})}}{e^{j\Omega/2} - e^{-j\Omega/2}} = \sin(\Omega(S+\frac{1}{2})) \sin(\Omega/2)
\]

Moments

Similar to CT, the value of X(\Omega) at \Omega = 0 is the sum of x[n] over time t.

\[
X(0) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x[n]
\]

As the function widens in n the Fourier transform narrows in Ω.
Moments
The value of \( x(0) \) is the integral of \( X(\omega) \) divided by \( 2\pi \).
\[
x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega
\]
\( x_1(t) \)
\[
X_1(\omega) = \frac{2\sin \omega}{\omega}
\]
\[
t = 0
\]

Stretching Time
Stretching time compresses frequency and increases amplitude (preserving area). 
\( x_1(t) \)
\[
X_1(\omega) = \frac{2\sin \omega}{\omega}
\]

Compressing Time to the Limit
Alternatively, we could compress time while keeping area = 1.
\[ x(t) \]
\[
X(\omega) = \frac{\sin \omega/2}{\omega/2}
\]
In the limit, the pulse has zero width but area 1!

Math With Impulses
Although physically unrealizable, the impulse (a.k.a. Dirac delta) function is useful as a mathematically tractable approximation to a very brief signal.

Example 1: Find the Fourier transform of a unit impulse function.
\[
X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{j\omega t} dt
\]
Since \( \delta(t) \) is zero except near \( t=0 \), only values of \( e^{j\omega t} \) near \( t=0 \) are important. Because \( e^{-j\omega t} \) is a smooth function of \( t \), \( e^{-j\omega t} \) can be replaced by \( e^{-j\omega t} \).
\[
X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1
\]
This matches our previous result which was based explicitly on a limit. Here the limit is implicit.

Example 2: Find the function whose Fourier transform is a unit impulse.
\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}
\]
Notice the similarity to the previous result:
\[
\delta(t) \quad \text{CTFT:} \quad 1
\]
These relations are duals of each other.
- A constant in time consists of a single frequency at \( \omega = 0 \).
- An impulse in time contains components at all frequencies.

Math With Impulses
Delta functions are similarly useful in discrete-time transforms.

Let
\[
X(\Omega) = \sum_{m=-\infty}^{\infty} \delta(\Omega - 2\pi m)
\]
where the sum results because DT Fourier Transforms are periodic in \( 2\pi \). Then
\[
x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega = \frac{1}{2\pi} \int_{2\pi} \delta(\Omega) e^{j\Omega n} d\Omega = \frac{1}{2\pi} \int_{2\pi} \delta(\Omega) d\Omega = \frac{1}{2\pi}
\]
Thus if \( x[n] = 1 \) for all \( n \), the transform is a delta function in frequency.
\[
1 \quad \text{DTFT:} \quad \sum_{m=-\infty}^{\infty} 2\pi \delta(\Omega - 2\pi m)
\]
We previously showed a similar result for CT:
\[
1 \quad \text{CTFT:} \quad 2\pi \delta(\omega)
\]

Moments

Stretching Time

Compressing Time to the Limit

Math With Impulses

Math With Impulses

Moments

Stretching Time

Compressing Time to the Limit

Math With Impulses

Math With Impulses
Using delta functions to calculate the transforms of complex exponentials.

Let

\[ X(\omega) = \delta(\omega - \omega_0) \]

then

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0)e^{j\omega t}d\omega \]

The delta function is zero except when \( \omega = \omega_0 \). Therefore we can substitute \( e^{j\omega t} \) for \( \delta(\omega - \omega_0) \) without changing the result of the integration.

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0)e^{j\omega t}d\omega = \frac{1}{2\pi} e^{j\omega_0 t} \int_{-\infty}^{\infty} \delta(\omega - \omega_0)d\omega = \frac{1}{2\pi} e^{j\omega_0 t} \]

Thus the transform of a complex exponential is a delta at that frequency.

\[ e^{j\omega_0 t} \overset{\text{CTFT}}{\rightarrow} 2\pi \delta(\omega - \omega_0) \]

The analogous expression for DT holds using analogous reasoning.

\[ e^{j\Omega_0 n} \overset{\text{DTFT}}{\rightarrow} 2\pi \delta(\omega - 2\pi \Omega_0) \]

### Relations Between Fourier Series and Fourier Transforms

### Continuous Time:

\[ e^{j2\pi k T t} \overset{\text{CTFT}}{\rightarrow} 2\pi \delta(\omega - \frac{2\pi k}{T}) \]

\[ x(t) = x(t+T) \overset{\text{CTFT}}{\rightarrow} X[k] \]

\[ x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} X[k]e^{j2\pi k T t} \overset{\text{CTFT}}{\rightarrow} \sum_{k=-\infty}^{\infty} 2\pi X[k] \delta(\omega - \frac{2\pi k}{T}) \]

### Discrete Time:

\[ e^{j2\pi k n} \overset{\text{DTFT}}{\rightarrow} \sum_{m=-\infty}^{\infty} 2\pi \delta(\Omega - \frac{2\pi k}{N} - 2\pi m) \]

\[ x[n] = x[n+N] \overset{\text{DTFT}}{\rightarrow} X[k] \]

\[ x[n] = x[n+N] = \sum_{k=(N)}^{\infty} X[k]e^{j2\pi k n} \overset{\text{DTFT}}{\rightarrow} \sum_{k=(N)}^{\infty} 2\pi X[k] \delta(\Omega - \frac{2\pi k}{N} - 2\pi m) \]

### Summary

**Discrete-Time Fourier Transform**

- Definition
- Examples
- Properties
- Relations between Fourier series and Fourier transforms (DT and CT)

**Quiz 1:** Tuesday, March 3, 2-4pm in 32-141.

- Coverage up to and including all of week 4.
- Closed book except for one page of notes (8.5"x11" both sides).
- No electronic devices. (No headphones, cellphones, calculators, ...)

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**Relations Between Fourier Transform and Fourier Series**

Each term in the Fourier series is replaced by an impulse in \( \omega \).