6.003: Signal Processing

Continuous-Time Fourier Transform

- Definition
- Examples
- Properties

Announcements:
- PSet 3 check-in is due today.
- No PSet 4 – practice quiz questions have been posted instead.
- Office Hours this Thursday (7-9pm) and Sunday (4-6pm)
  - practice quiz problems and general questions
- Solutions to practice quiz questions will be posted Sunday at 6pm
- Quiz 1: Tuesday, March 3, 2-4pm in 32-141.
  - Coverage up to and including all of week 4.
  - Closed book except for one page of notes (8.5”x11” both sides).
  - No electronic devices. (No headphones, cellphones, calculators, ...)

February 25, 2020
From Periodic to Aperiodic

Previously we have focused on Fourier representations of periodic signals: e.g., sounds, waves, music, ...

However, most real-world signals are not periodic. None are truly periodic since they do not have infinite duration!

Today: generalizing Fourier representations to include aperiodic signals.
Fourier Representations of Aperiodic Signals

How can we represent an aperiodic signal as a sum of sinusoids?

![Graph showing a square wave](image)
Fourier Representations of Aperiodic Signals

How can we represent an aperiodic signal as a sum of sinusoids?

Strategy: make a periodic version of $x(t)$ by summing shifted copies:

$$x_p(t) = \sum_{m=-\infty}^{\infty} x(t - mT)$$

Since $x_p(t)$ is periodic, it has a Fourier series (which depends on $T$).

Find Fourier series coefficients $X_p[k]$ and take the limit of $X_p[k]$ as $T \to \infty$.

As $T \to \infty$, $x_p(t) \to x(t)$ and Fourier series will approach Fourier transform.
Fourier Representations of Aperiodic Signals

Example.

Strategy: make a periodic version of \( x(t) \) by summing shifted copies:

\[
x_p(t) = \sum_{m=-\infty}^{\infty} x(t - mT)
\]

Calculate the Fourier series coefficients \( X_p[k] \):

\[
X_p[k] = \frac{1}{T} \int_{-S}^{S} e^{-j \frac{2\pi}{T} k t} \, dt = \frac{1}{T} \left. \frac{e^{-j \frac{2\pi}{T} k t}}{-j \frac{2\pi}{T} k} \right|_{-S}^{S} = \frac{2 \sin \left( \frac{2\pi k}{T} S \right)}{T \left( \frac{2\pi k}{T} \right)}
\]
Fourier Representations of Aperiodic Signals

Calculate the Fourier series coefficients $X_p[k]$:

$$X_p[k] = \frac{1}{T} \int_{-S}^{S} e^{-j\frac{2\pi}{T}kt} dt = \frac{1}{T} \left[ e^{-j\frac{2\pi}{T}kt} \right]_{-S}^{S} = \frac{2 \sin \left( \frac{2\pi k}{T} S \right)}{T \left( \frac{2\pi k}{T} \right)}$$

Plot the resulting Fourier coefficients when $S=1$ and $T=8$.

What happens if you double the period $T$?
Fourier Representations of Aperiodic Signals

Calculate the Fourier series coefficients $X_p[k]$: 

$$X_p[k] = \frac{1}{T} \int_{-S}^{S} e^{-j\frac{2\pi}{T}kt} dt = \frac{1}{T} \left| e^{-j\frac{2\pi}{T}kt} \right|_{-S}^{S} = \frac{2 \sin \left(\frac{2\pi k}{T} S\right)}{T \left(\frac{2\pi k}{T}\right)}$$

Plot the resulting Fourier coefficients when $S=1$ and $T=8$.

What happens if you double the period $T$?

- The amplitude will shrink by a factor of 2.
- There will be twice as many samples per period of the sin function.

The red samples are at new intermediate frequencies.
Fourier Representations of Aperiodic Signals

Define a new function \( X(\omega) \) where \( \omega = \frac{2\pi k}{T} \).

\[
TX_p[k] = \frac{2 \sin \left( \frac{2\pi k S}{T} \right)}{2\pi k} = 2 \frac{\sin(\omega S)}{\omega} = X(\omega)
\]

Then \( TX_p[k] \) represents samples of \( X(\omega) \) with increasing resolution in \( \omega \).

\[
TX_p[k] = X(\omega)
\]

\( S=1 \) and \( T=8 \):

\( \omega = \frac{2\pi k}{T} \)

\( S=1 \) and \( T=16 \):

\( \omega = \frac{2\pi k}{T} \)

\( S=1 \) and \( T=32 \):

\( \omega = \frac{2\pi k}{T} \)

The discrete function \( TX_p[k] \) is a sampled version of the function \( X(\Omega) \).
Fourier Representations of Aperiodic Signals

We can reconstruct $x(t)$ from $X(\omega)$ using Riemann sums.

$$x_p(t) = \sum_{k=-\infty}^{\infty} X_p[k] e^{\frac{j2\pi}{T}kt} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} TX_p[k] e^{\frac{j2\pi}{T}kt} \left( \frac{2\pi}{T} \right)$$

$$x(t) = \lim_{T \to \infty} x_p(t) = \lim_{T \to \infty} \frac{1}{2\pi} \sum_{k} TX_p[k] e^{\frac{j2\pi}{T}kt} \left( \frac{2\pi}{T} \right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$TX_p[k] = X(\omega)$

$S=1$ and $T=8$: $\omega = \frac{2\pi k}{T}$

$S=1$ and $T=16$: $\omega = \frac{2\pi k}{T}$

$S=1$ and $T=32$: $\omega = \frac{2\pi k}{T}$

Fourier Transform relation: $x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$
Continuous-Time Fourier Representations

Fourier series and transforms are similar: both represent signals by their frequency content.

Continuous-Time Fourier Series

\[ X[k] = \frac{1}{T} \int_T x(t)e^{-jk\omega_0 t} dt \]

analysis equation

\[ x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t} \]
synthesis equation

where \( \omega_0 = \frac{2\pi}{T} \)

Continuous-Time Fourier Transform

\[ X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \]

analysis equation

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega \]
synthesis equation
Continuous-Time Fourier Representations

All of the information in a periodic signal is contained in one period. The information in an aperiodic signal is spread across all time.

Continuous-Time Fourier Series

$$X[k] = \frac{1}{T} \int_{T} x(t)e^{-jk\omega_{o}t} \, dt$$  \hspace{1cm} \text{analysis equation}

$$x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_{o}t}$$  \hspace{1cm} \text{synthesis equation}

where $\omega_{o} = \frac{2\pi}{T}$

Continuous-Time Fourier Transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} \, dt$$  \hspace{1cm} \text{analysis equation}

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} \, d\omega$$  \hspace{1cm} \text{synthesis equation}
Continuous-Time Fourier Representations

Periodic signals can be synthesized from a discrete set of harmonics. Aperiodic signals generally require all possible frequencies.

Continuous-Time Fourier Series

\[
X[k] = \frac{1}{T} \int_T x(t)e^{-jk\omega_0 t} \, dt
\]

analysis equation

\[
x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}
\]
synthesis equation

where \( \omega_0 = \frac{2\pi}{T} \)

Continuous-Time Fourier Transform

\[
X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} \, dt
\]

analysis equation

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} \, d\omega
\]
synthesis equation
Continuous-Time Fourier Representations

Harmonic frequencies $k\omega_o$ are samples of continuous frequency $\omega$.

Continuous-Time Fourier Series

\[
X[k] = \frac{1}{T} \int_T x(t)e^{-jk\omega_0t} \, dt
\]

\[
x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0t}
\]

where $\omega_0 = \frac{2\pi}{T}$

Continuous-Time Fourier Transform

\[
X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} \, dt
\]

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} \, d\omega
\]
Examples of Fourier Transforms

Find the Fourier Transform (FT) of a rectangular pulse:

\[ x(t) = \begin{cases} 
1 & -1 < t < 1 \\
0 & \text{otherwise}
\end{cases} \]

\[ X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-1}^{1} e^{-j\omega t} dt = \frac{e^{-j\omega t}}{-j\omega} \bigg|_{-1}^{1} = 2 \frac{\sin \omega}{\omega} \]

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \]

\( X(\omega) \) provides a recipe for constructing \( x(t) \) from sinusoidal components:

A square pulse contains (almost) all frequencies \( \omega \) (missing just \( \pi, 2\pi, \ldots \)).
The Fourier transform of a rectangular pulse is \( 2 \frac{\sin \omega}{\omega} \).

\[ x(t) \quad \overset{\text{FT}}{\Rightarrow} \quad X(\omega) \]

\( X(\omega) \) contains all frequencies \( \omega \) except non-zero multiples of \( \pi \).

Why is the transform zero at non-zero multiples of \( \pi \)?
What is special about those frequencies?
Why isn’t \( X(\omega) \) zero at \( \omega = 0 \)?
Check Yourself

If $\omega = m\pi$, there are exactly $m$ periods of $e^{-j\omega t}$ in $-1 < t < 1$.

$$X(\omega = m\pi) = \int_{-1}^{1} e^{-j\omega t} dt = \int_{-1}^{1} e^{-jm\pi t} dt = \begin{cases} 2 & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases}$$

No Fourier components are needed for frequencies $\omega = m\pi$. However DC is required to offset $x(t)$ so that $x(t) \geq 0$ for all $t$. 
Examples of Fourier Transforms

Find the Fourier Transform of a delayed rectangular pulse:

\[ x_d(t) = \begin{cases} 
1 & 0 < t < 2 \\
0 & \text{otherwise}
\end{cases} \]

\[ x_d(t) \]

\[ \cdots \]

\[ \begin{array}{c}
1 \\
0 \\
2 \\
\end{array} \]

\[ t \]

\[ X_d(\omega) = \int_{-\infty}^{\infty} x_d(t)e^{-j\omega t} dt = \int_{0}^{2} e^{-j\omega t} dt = \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{0}^{2} = \frac{1}{j\omega}(1 - e^{-j2\omega}) \]

We can see the relation to \( X(\omega) \) by factoring out \( e^{-j\omega} \):

\[ X_d(\omega) = \frac{1}{j\omega} e^{-j\omega}(e^{j\omega} - e^{-j\omega}) = e^{-j\omega} \left( 2 \frac{\sin \omega}{\omega} \right) = e^{-j\omega} X(\omega) \]
Properties of Fourier Transforms

Time delay maps to linear phase delay of the Fourier transform.

If \( x(t) \xlongleftrightarrow{\text{FT}} X(\omega) \)

then \( x(t - \tau) \xlongleftrightarrow{\text{FT}} e^{-j\omega\tau} X(\omega) \)

\[
\begin{align*}
X(\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\
Y(\omega) &= \int_{-\infty}^{\infty} x(t - \tau) e^{-j\omega t} dt
\end{align*}
\]

Let \( u = t - \tau \) (and therefore \( du = dt \) since \( \tau \) is a constant)

\[
\begin{align*}
Y(\omega) &= \int_{-\infty}^{\infty} x(u) e^{-j\omega(u + \tau)} du = e^{-j\omega\tau} \int_{-\infty}^{\infty} x(u) e^{-j\omega u} du = e^{-j\omega\tau} X(\omega)
\end{align*}
\]

Why does time delay change phase by an amount proportional to frequency?
Why does time delay change phase by an amount proportional to frequency?

Just enough phase to delay every frequency component by \( t=1 \).

The same amount of time corresponds to different amounts of phase.
Fourier Transform

Scaling time.

Consider the following signal and its Fourier transform.

Time representation:

Frequency representation:

$$X_1(\omega) = \frac{2 \sin \omega}{\omega}$$

How would these scale if time were stretched?
Signal $x_2(t)$ and its Fourier transform $X_2(\omega)$ are shown below.

Which of the following is true?

1. $b = 2$ and $\omega_0 = \pi/2$
2. $b = 2$ and $\omega_0 = 2\pi$
3. $b = 4$ and $\omega_0 = \pi/2$
4. $b = 4$ and $\omega_0 = 2\pi$
5. none of the above
Check Yourself

Find the Fourier transform.

$$X_2(\omega) = \int_{-2}^{2} e^{-j\omega t} \, dt = \frac{e^{-j\omega t}}{-j\omega} \bigg|_{-2}^{2} = \frac{2 \sin 2\omega}{\omega} = \frac{4 \sin 2\omega}{2\omega}$$

Stretching time compresses frequency.
Signal $x_2(t)$ and its Fourier transform $X_2(\omega)$ are shown below.

Which of the following is true?  

1. $b = 2$ and $\omega_0 = \pi/2$
2. $b = 2$ and $\omega_0 = 2\pi$
3. $b = 4$ and $\omega_0 = \pi/2$
4. $b = 4$ and $\omega_0 = 2\pi$
5. none of the above
Fourier Transforms

Find a general scaling rule.

Let $x_2(t) = x_1(at)$ where $a > 0$.

$$X_2(\omega) = \int_{-\infty}^{\infty} x_1(at)e^{-j\omega t} dt$$

Let $\tau = at$. Then $d\tau = a \, dt$.

$$X_2(\omega) = \int_{-\infty}^{\infty} x_1(\tau)e^{-j\omega \tau/a} \frac{1}{a} d\tau = \frac{1}{a} X_1 \left( \frac{\omega}{a} \right)$$

Stretching time compresses frequency and increases amplitude (preserving area).
Moments

The value of $X(\omega)$ at $\omega = 0$ is the integral of $x(t)$ over time $t$.

$$X(\omega)|_{\omega=0} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt = \int_{-\infty}^{\infty} x(t) e^{-j0t} \, dt = \int_{-\infty}^{\infty} x(t) \, dt$$

$x_1(t)$

area = 2

$X_1(\omega) = \frac{2 \sin \omega}{\omega}$
Moments

The value of $x(0)$ is the integral of $X(\omega)$ divided by $2\pi$.

$$x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) d\omega$$

$$X_1(\omega) = \frac{2 \sin \omega}{\omega}$$

Area $\frac{1}{2\pi} = 1$
Moments

The value of $x(0)$ is the integral of $X(\omega)$ divided by $2\pi$.

$$x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) d\omega$$

$x_1(t)$

$X_1(\omega) = \frac{2 \sin \omega}{\omega}$

The area $\frac{2\pi}{2\pi} = 1$ and equal areas!
Stretching Time

Stretching time compresses frequency and increases amplitude (preserving area).

\[ x_1(t) = \begin{cases} 1 & \text{for } t \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \]

\[ X_1(\omega) = \frac{2 \sin \omega}{\omega} \]

\[ X_1(\omega) = \begin{cases} 2 & \text{for } \omega = 0 \\ 4 & \text{for } \omega = \pi \end{cases} \]
**Compressing Time to the Limit**

Alternatively, we could compress time while keeping area $= 1$.

In the limit, the pulse has zero width but area 1!

We represent this limit with the delta function: $\delta(t)$.
Math With Impulses

Although physically unrealizable, the impulse (a.k.a. Dirac delta) function is useful as a mathematically tractable approximation to a very brief signal.

Example 1: Find the Fourier transform of a unit impulse function.

\[ X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt \]

Since \( \delta(t) \) is zero except near \( t=0 \), only values of \( e^{-j\omega t} \) near \( t=0 \) are important. Because \( e^{-j\omega t} \) is a smooth function of \( t \), \( e^{-j\omega t} \) can be replaced by \( e^{-j\omega 0} \):

\[ X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega 0} dt = 1 \]

This matches our previous result which was based explicitly on a limit. Here the limit is implicit.
Math With Impulses

Although physically unrealizable, the impulse function is extremely useful as a mathematically tractable approximation to a very brief signal.

Example 2: Find the function whose Fourier transform is a unit impulse.

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j0t} d\omega = \frac{1}{2\pi}
\]

\[
1 \overset{\text{CTFT}}{\iff} 2\pi \delta(\omega)
\]

Notice the similarity to the previous result:

\[
\delta(t) \overset{\text{CTFT}}{\iff} 1
\]

These relations are **duals** of each other.

- A constant in time consists of a single frequency at \( \omega = 0 \).
- An impulse in time contains components at all frequencies.
Duality

The continuous-time Fourier transform and its inverse are symmetric except for the minus sign in the exponential and the factor of $2\pi$.

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

If $x(t) \xrightarrow{\text{FT}} X(\omega)$
then $X(t) \xrightarrow{\text{FT}} 2\pi x(-\omega)$

$$x(t) = \frac{1}{2\pi} \int X(\omega) e^{j\omega t} d\omega \quad \text{(FT synthesis)}$$

$$x(\omega) = \frac{1}{2\pi} \int X(t) e^{j\omega t} dt \quad \text{(swap } t \text{ and } \omega)$$

$$x(-\omega) = \frac{1}{2\pi} \int X(t) e^{-j\omega t} dt \quad \text{(change sign of } \omega)$$

$$2\pi x(-\omega) = \int X(t) e^{-j\omega t} dt \quad \text{(multiply by } 2\pi)$$
Duality

Graphic representation of duality.

If \( x(t) \overset{\text{FT}}{\leftrightarrow} X(\omega) \)
then \( X(t) \overset{\text{FT}}{\leftrightarrow} 2\pi x(-\omega) \)

\[
x(t) \overset{\text{FT}}{\leftrightarrow} X(\omega) \\
\omega \rightarrow t \\
t \rightarrow -\omega \text{ and scale up by } 2\pi \\
X(t) \overset{\text{FT}}{\leftrightarrow} 2\pi x(-\omega)
\]
Duality

Using duality to find new transform pairs.

\[ x(t) \overset{FT}{\leftrightarrow} X(\omega) \]

\[ \omega \rightarrow t \quad t \rightarrow -\omega \text{ and scale up by } 2\pi \]

\[ X(t) \overset{FT}{\leftrightarrow} 2\pi x(-\omega) \]

\[ x(t) = \delta(t) \]

\[ X(\omega) = 1 \]

\[ 2\pi x(-\omega) = 2\pi \delta(\omega) \]

Fourier transform of a constant in time is an impulse in frequency.
Complex Exponentials

Using duality to find the Fourier transform of a complex exponential.

\[ x(t) \overset{FT}{\leftrightarrow} X(\omega) \]

\[ \omega \rightarrow t \quad t \rightarrow -\omega \text{ and scale up by } 2\pi \]

\[ X(t) \overset{FT}{\leftrightarrow} 2\pi x(-\omega) \]

\[ \delta(t) \overset{FT}{\leftrightarrow} 1 \quad \text{(limiting case of narrow pulse)} \]

\[ \delta(t + T) \overset{FT}{\leftrightarrow} e^{j\omega T} \quad \text{(delay property)} \]

\[ \omega \rightarrow t \quad t \rightarrow -\omega \text{ and scale up by } 2\pi \]

\[ e^{jtT} \overset{FT}{\leftrightarrow} 2\pi \delta(-\omega + T) = 2\pi \delta(\omega - T) \]

\( T \rightarrow \omega_0 \)

\[ e^{j\omega_0 t} \overset{FT}{\leftrightarrow} 2\pi \delta(\omega - \omega_0) \]
Complex Exponentials

Using duality to find the Fourier transform of a complex exponential.

\[ e^{j\omega_0 t} \overset{\text{FT}}{\leftrightarrow} 2\pi \delta(\omega - \omega_o) \]

\[ x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi}{T}kt} \quad \text{CTFS} \quad \overset{\leftrightarrow}{\rightarrow} \quad X[k] \]

\[ x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi}{T}kt} \quad \text{CTFT} \quad \overset{\leftrightarrow}{\leftrightarrow} \quad \sum_{k=-\infty}^{\infty} 2\pi X[k] \delta\left(\omega - \frac{2\pi}{T}k\right) \]
Relation between Fourier Transform and Fourier Series

Each term in the Fourier series is replaced by an impulse in $\omega$.

$$x(t) = \sum_{k=-\infty}^{\infty} x_p(t - kT)$$

$$X[k]$$

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi X[k] \delta(\omega - k\frac{2\pi}{T})$$

$$\omega$$
Summary: Continuous-Time Fourier Transform (CTFT)

Definition
- analysis and synthesis relations: analogous to CTFS

Examples
- square pulse

Properties
- time delay
- time scaling
- moment relations

Next time: Fourier Transform for discrete-time signals.