February 4, 2020

**6.003: Signal Processing**

**Signal Processing**
- Overview of Subject
- Signals: Definitions, Examples, and Operations
- Time and Frequency Representations
- Fourier Series

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**Signals**
- Functions that contain and convey information.
- May have 1 or 2 or 3 or even more independent variables.

**Course Mechanics**

**Schedule**
- **Lecture:** Tue. and Thu. 2-3pm in 32-141
- **Recitation:** Tue. and Thu. 3-4pm in 24-121 or 26-328
- **Office Hours:** Tue. and Thu. 4-5pm in 24-121 or 26-328
  Wed. and Thu. 7-9pm in 36-144
  Sun. 4-6pm (room TBD)

**Homework** – issued Tuesdays, due following Tuesday at noon
- **Drills:** focus on facts, definitions, and simple concepts
  - online with immediate feedback (not graded)
- **Problems:** focus on developing problem solving skills
  - pencil and paper problems taken from previous exams
  - simple computational extensions to real-world data
  - completely specified, unambiguous, self-contained
- **Labs:** focus on applications of 6.003 to authentic problems
  - more open-ended, multiple approaches, multiple solutions
  - deepen understanding and demonstrate wide applicability
  - issued Tuesday, required check-in Thursday, due following Tuesday

Two Midterms and a Final Exam

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**Signal Processing**
- Signals are functions that contain and convey information.
- Examples:
  - the MP3 representation of a sound
  - the JPEG representation of a picture
  - an MRI image of a brain

**Signal Processing** develops the use of signals as abstractions:
- identifying signals in physical, mathematical, computation contexts,
- analyzing signals to understand the information they contain, and
- manipulating signals to modify and/or extract information.

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**Signals**
- Functions that are used to convey information.
- May have 1 or 2 or 3 or even more independent variables.
- **dependent variable** can be a scalar or a vector

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**Classical analyses** use a variety of maths, especially calculus. We will also use computation to solve real-world problems that are difficult or impossible to solve analytically.

→ strengthens ties to the real world
Signals are functions that are used to convey information.
- **Dependent variable** can be real, imaginary, or complex-valued.

\[ x(t) = e^{j2\pi t} = \cos 2\pi t + j\sin 2\pi t \]

Signals from physical systems are often of **continuous** domain:
- Continuous time – measured in seconds
- Continuous spatial coordinates – measured in meters

Computations usually manipulate functions of **discrete** domain:
- Discrete time – measured in samples
- Discrete spatial coordinates – measured in samples

Relating continuous and discrete representations enables application of computational methods to solve problems that are intrinsically continuous.

**Sampling:** converting CT signals to DT

\[ x[n] = x(nT) \]

\[ T = \text{sampling interval} \]

Important for computational manipulation of physical data.
- Digital representations of audio signals (as in MP3)
- Digital representations of images (as in JPEG)

**Reconstruction:** converting DT signals to CT

- **Zero-order hold**

\[ x(t) = x(t + T) \]

\[ T = \text{sampling interval} \]

Commonly used in audio output devices

- **Piecewise linear**

\[ x[n] = x[n + N] \]

\[ T = \text{sampling interval} \]

Commonly used in rendering images

**Periodic signals** consist of repeated cycles (periods).

- **Periodic**

\[ x(t) = x(t + T) \]

- **Aperiodic**

\[ x(t) \]

Useful for modeling periodic or nearly-periodic systems
- Planetary motions
- Vibrating strings
Signals

Right-sided signals are zero before some starting time. Left-sided signals are zero after some ending time.

\[
\begin{align*}
\text{right-sided} & : x(t) \\
\text{left-sided} & : x(t) \\
\end{align*}
\]

Useful for modeling systems that have a well-defined starting point:
- piano note
- striking a cymbal

Musical Sounds as Signals

Even though these sounds have the same pitch, they sound different.

\[
\begin{align*}
\text{piano} & : \cos(2\pi \omega o t) \\
\text{cello} & : \sin(2\pi \omega o t) \\
\text{bassoon} & : \cos(2\pi \omega o t) + d_1 \sin(2\pi \omega o t) \\
\text{ocean} & : \cos(2\pi \omega o t) + d_2 \sin(2\pi \omega o t) \\
\text{horn} & : \sin(2\pi \omega o t) \\
\text{altosax} & : \sin(2\pi \omega o t) \\
\text{violin} & : \cos(2\pi \omega o t) + d_3 \sin(2\pi \omega o t) \\
\end{align*}
\]

It’s not clear how the differences relate to properties of the signals.
(audio clips from http://theremin.music.uiowa.edu)

Musical Signals as Sums of Sinusoids

One way to characterize differences between these signals is express each as a sum of sinusoids.

\[
f(t) = \sum_{k=0}^{\infty} (r_k \cos(k\omega o t) + d_k \sin(k\omega o t))
\]

Since these sounds are (nearly) periodic, the frequencies of the dominant sinusoids are (nearly) integer multiples of a fundamental frequency \( \omega o \).
Harmonic Structure
The sum of sinusoids describes the distribution of energy across frequencies.

\[ f(t) = \sum_{k=0}^{\infty} (c_k \cos k\omega_0 t + d_k \sin k\omega_0 t) = \sum_{k=0}^{\infty} m_k \cos (k\omega_0 t + \phi_k) \]

where \( m_k^2 = c_k^2 + d_k^2 \) and \( \tan \phi_k = \frac{d_k}{c_k} \).

This distribution represents the harmonic structure of the signal.

Consonance and Dissonance
Which of the following pairs is least consonant?

<table>
<thead>
<tr>
<th>A1</th>
<th>A2</th>
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<tbody>
<tr>
<td><img src="A1_waveform" alt="Waveform" /></td>
<td><img src="A2_waveform" alt="Waveform" /></td>
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<td>B1</td>
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<td>C2</td>
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<tr>
<td><img src="C1_waveform" alt="Waveform" /></td>
<td><img src="C2_waveform" alt="Waveform" /></td>
</tr>
</tbody>
</table>

Obvious from the sounds ... less obvious from the waveforms.

Express Each Signal as a Sum of Sinusoids

\[ f(t) = \sum_{k=0}^{\infty} m_k \cos (k\omega_0 t + \phi_k) \]

\[ = m_1 \cos (\omega_0 t + \phi_1) + m_2 \cos (2\omega_0 t + \phi_2) + m_3 \cos (3\omega_0 t + \phi_3) + \cdots \]

The signal \( f(t) \) can be expressed as a discrete set of frequency components:

\[ \omega_0: \ m_1, \ \phi_1 \]

\[ 2\omega_0: \ m_2, \ \phi_2 \]

\[ 3\omega_0: \ m_3, \ \phi_3 \]

\cdots

Musical Sounds as Signals

Time functions do a poor job of conveying consonance and dissonance.

octave (D+D')

\[ \begin{align*}
\text{freq} & \quad \text{time} \\
0 & \quad 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \\
1 & \quad 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \\
-1 & \quad 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \
\end{align*} \]

D'  |  A  |  E#
\[ \begin{align*}
\text{time (periods of "D")} & \\
0 & \quad 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \\
1 & \quad 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \\
-1 & \quad 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \\
\end{align*} \]

Harmonic structure conveys consonance and dissonance better.
Q1: Under what conditions can we write \( f(t) \) as a Fourier series?

Q2: How do we find the coefficients \( c_k \) and \( d_k \)?

All harmonics of \( \omega_0 \) (\( \cos(k\omega_0 t) \) or \( \sin(k\omega_0 t) \)) are periodic in \( T = 2\pi/\omega_0 \).

\( \rightarrow \) all sums of such signals are periodic in \( T = 2\pi/\omega_0 \).

\( \rightarrow \) Fourier series can only represent periodic signals.

Calculating Fourier Coefficients

Isolate the \( c_1 \) term by multiplying both sides by \( \cos(\omega_0 t) \) before integrating.

\[
\begin{align*}
f(t) &= f(t+T) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)) \\
\int_0^T f(t) \cos(\omega_0 t) dt &= \int_0^T c_0 \cos(\omega_0 t) dt \\
&\quad + \sum_{k=1}^{\infty} \int_0^T c_k \left( \frac{1}{2} \cos((k-1)\omega_0 t) + \frac{1}{2} \cos((k+1)\omega_0 t) \right) dt \\
&\quad + \sum_{k=1}^{\infty} \int_0^T d_k \left( \frac{1}{2} \sin((k-1)\omega_0 t) + \frac{1}{2} \sin((k+1)\omega_0 t) \right) dt
\end{align*}
\]

If \( k = l \), then \( \sin((k-l)\omega_0 t) = 0 \) and the integral is 0.

All of the other \( d_k \) terms are harmonic sinusoids that integrate to 0.

The only non-zero term on the right side is \( \frac{T}{2} c_1 \).

We can solve to get an expression for \( c_1 \) as

\[
c_1 = \frac{2}{T} \int_0^T f(t) \cos(\omega_0 t) dt
\]
Calculating Fourier Coefficients

Summarizing...

If \( f(t) \) is expressed as a Fourier series
\[
f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t))
\]
the Fourier coefficients are given by
\[
c_0 = \frac{1}{T} \int_T f(t) \, dt
\]
\[
c_k = \frac{2}{T} \int_T f(t) \cos(k\omega_0 t) \, dt; \quad k = 1, 2, 3, \ldots
\]
\[
d_k = \frac{2}{T} \int_T f(t) \sin(k\omega_0 t) \, dt; \quad k = 1, 2, 3, \ldots
\]

Example of Synthesis

Generate \( f(t) \) from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients
\[
f(t) = c_0 - \sum_{k=1}^{\infty} (c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)) = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)
\]
\[
f(t) = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)
\]

Example of Analysis

Find the Fourier series coefficients for the following triangle wave:
\[
f(t) = \begin{cases} 1 & \text{for } 0 \leq t < 1 \\ 0 & \text{for } 1 \leq t < 2 \\ 1 & \text{for } 2 \leq t < 3 \\ 0 & \text{for } 3 \leq t < 4 \end{cases}
\]

Two Views of the Same Signal

The harmonic expansion provides an alternative view of the signal.
\[
f(t) = \sum_{k=0}^{\infty} (c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)) = \sum_{k=0}^{\infty} m_k \cos(k\omega_0 t + \phi_k)
\]

We can view the musical signal as
- a function of time \( f(t) \), or
- as a sum of harmonics with amplitudes \( m_k \) and phase angles \( \phi_k \).

Both views are useful. For example,
- the peak sound pressure is more easily seen in \( f(t) \), while
- consonance is more easily analyzed by comparing harmonics.

This type of harmonic analysis is an example of Fourier Analysis, which is a major theme of this subject.

Next Time: understanding Fourier series and their properties.