Discrete Cosine Transform

\[ X_C[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \cos \left( \frac{\pi k}{N} \left( n + \frac{1}{2} \right) \right) \]

\[ x[n] = X_C[0] + 2 \sum_{k=1}^{N-1} X_C[k] \cos \left( \frac{\pi k}{N} \left( n + \frac{1}{2} \right) \right) \]
Motivation

The Discrete Fourier Transform (DFT) implicitly represents the frequencies that are contained in a periodically extended version of the input signal, and periodic extension can generate frequencies that are not present in the original signal.

Consider the following $8 \times 8$ example.

The brightnesses of left and right edges are different and will generate a sequence of large transitions when periodically extended.
Consider a single row from the previous image. The DFT implicitly extends the signal (here a ramp) periodically.

\[ x[n] = x[n + 8] \]

Although the function is smooth from \( n = 0 \) to 7, the periodic extension contains a series of steps.
Motivation: 1D

We can eliminate the step discontinuities by first replicating one period in reverse order and then extending the result periodically.

\[ y[n] = y[n + 16] \]

The resulting signal is continuous across the edges (however the slope is still discontinuous).
Motivation: 1D

Finally, insert zeros between successive samples.

$$z[n] = z[n + 32]$$

The resulting signal is real-valued, symmetric about $n = 0$, periodic in $4N$, and contains only odd numbered samples.

The DFT of this signal is real, symmetric about $k = 0$ and anti-periodic. It is completely characterized by $N$ values: $Z[0]$ to $Z[7]$.

This process is captured in the **Discrete Cosine Transform**.
Discrete Cosine Transform

The Discrete Cosine Transform (DCT) is described by analysis and synthesis equations that are analogous to those of the DFT.

\[ X_C[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \cos \left( \frac{\pi k}{N} \left( n + \frac{1}{2} \right) \right) \]  
(analysis)

\[ x[n] = X_C[0] + 2 \sum_{k=1}^{N-1} X_C[k] \cos \left( \frac{\pi k}{N} \left( n + \frac{1}{2} \right) \right) \]  
(synthesis)
Comparison of DFT and DCT Basis Functions

DFT (real and imaginary parts) versus DCT.

\[
\begin{align*}
\text{Re}(e^{j\frac{2\pi k}{N} n}) & \quad \text{Im}(e^{j\frac{2\pi k}{N} n}) & \quad \cos\left(\frac{\pi k}{N} (n + \frac{1}{2})\right) \\
\end{align*}
\]

\[
\begin{array}{cccc}
k = 0 & n & n & n \\
\hline
k = 1 & n & n & n \\
\hline
k = 2 & n & n & n \\
\hline
k = 3 & n & n & n \\
\hline
k = 4 & n & n & n \\
\hline
k = 5 & n & n & n \\
\hline
k = 6 & n & n & n \\
\hline
k = 7 & n & n & n \\
\end{array}
\]
Much of the utility of Fourier transforms in general and the DFT in particular results from properties of the Fourier basis functions:

\[
\begin{align*}
\text{DTFT} & : \quad e^{j\Omega n} \\
\text{DTFS} & : \quad e^{j\frac{2\pi k}{N} n} \\
\text{DFT} & : \quad e^{j\frac{2\pi k}{N} n}
\end{align*}
\]

To better understand the DCT, we need to similarly understand its basis functions.

\[
\text{DCT} \quad \cos \left( \frac{\pi k}{N} \left( n + \frac{1}{2} \right) \right)
\]
DCT Basis Functions

The $k^{th}$ DCT basis function of order $N$ is given by

$$
\phi_k[n] = \cos \left( \frac{\pi k}{N} \left( n + \frac{1}{2} \right) \right).
$$

How many of the following symmetries are true?

- $\phi_k[n+2N] = \phi_k[n]$
- $\phi_k[n+N] = (-1)^k \phi_k[n]$
- $\phi_k[n-N] = (-1)^k \phi_k[n]$
- $\phi_k[(N-1)-n] = (-1)^k \phi_k[n]$
DCT Basis Functions

The $k^{th}$ DCT basis function of order $N$ is given by

$$\phi_k[n] = \cos \left( \frac{\pi k}{N} \left(n + \frac{1}{2}\right) \right).$$

The first property

$$\phi_k[n+2N] = \phi_k[n]$$

follows from the periodicity of the cosine function, as follows.

$$\phi_k[n+2N] = \cos \left( \frac{\pi k}{N} \left(2N+n+\frac{1}{2}\right) \right)$$

$$= \cos \left( \frac{\pi k}{N} 2N + \frac{\pi k}{N} \left(n+\frac{1}{2}\right) \right)$$

$$= \cos \left( 2\pi k + \frac{\pi k}{N} \left(n+\frac{1}{2}\right) \right)$$

$$= \cos \left( \frac{\pi k}{N} \left(n+\frac{1}{2}\right) \right) = \phi_k[n]$$

Notice that $\phi_k[n]$ is not periodic in $N$!
The $k^{th}$ DCT basis function of order $N$ is given by

$$\phi_k[n] = \cos \left( \frac{\pi k}{N} \left(n + \frac{1}{2}\right) \right).$$

The second property

$$\phi_k[n+N] = (-1)^k \phi_k[n]$$

addresses symmetry in $N$.

$$\phi_k[n+N] = \cos \left( \frac{\pi k}{N} (N+n+\frac{1}{2}) \right)$$

$$= \cos \left( \frac{\pi k}{N} N + \frac{\pi k}{N} \left(n + \frac{1}{2}\right) \right)$$

$$= \cos \left( \pi k + \frac{\pi k}{N} \left(n + \frac{1}{2}\right) \right)$$

$$= (-1)^k \cos \left( \frac{\pi k}{N} \left(n + \frac{1}{2}\right) \right) = (-1)^k \phi_k[n]$$
The $k^{th}$ DCT basis function of order $N$ is given by

$$\phi_k[n] = \cos \left( \frac{\pi k}{N} \left( n + \frac{1}{2} \right) \right).$$

The third property

$$\phi_k[n-N] = (-1)^k \phi_k[n]$$

follows from the first two.

$$\phi_k[n-N] = \phi_k[n+2N-N] = \phi_k[n+N] = (-1)^k \phi_k[n]$$
DCT Basis Functions

The $k^{th}$ DCT basis function of order $N$ is given by

$$\phi_k[n] = \cos \left( \frac{\pi k}{N} \left(n + \frac{1}{2}\right) \right).$$

The fourth property

$$\phi_k[(N-1)-n] = (-1)^k \phi_k[n]$$

describes symmetry about the point $n = (N-1)/2$.

$$\phi_k[(N-1)-n] = \cos \left( \frac{\pi k}{N} \left(N-1-n+\frac{1}{2}\right) \right)$$

$$= \cos \left( \frac{\pi k}{N} \left(N-\left(n+\frac{1}{2}\right)\right) \right)$$

$$= \cos(\pi k) \cos \left( \frac{\pi k}{N} \left(n+\frac{1}{2}\right) \right) + \sin(\pi k) \sin \left( \frac{\pi k}{N} \left(n+\frac{1}{2}\right) \right)$$

$$= (-1)^k \cos \left( \frac{\pi k}{N} \left(n+\frac{1}{2}\right) \right) = (-1)^k \phi_k[n]$$
We can use the previous properties to calculate useful facts.

Show that

\[
\sum_{n=0}^{N-1} \phi_k[n] = \sum_{n=0}^{N-1} \cos \left( \frac{\pi k}{N} \left( n + \frac{1}{2} \right) \right) = N \delta[k].
\]
DCT Basis Functions

Show that
\[ \sum_{n=0}^{N-1} \phi_k[n] = \sum_{n=0}^{N-1} \cos \left( \frac{\pi k}{N} (n + \frac{1}{2}) \right) = N \delta[k]. \]

If \( k \) is odd, then by property 4, the sum \( \sum_{n=0}^{N-1} \phi_k[n] \) zero.

If \( k \) is even, then we can replace \( k \) by \( 2l \) where \( l \) is an integer:
\[ \sum_{n=0}^{N-1} \phi_k[n] = \sum_{n=0}^{N-1} \cos \left( \frac{2\pi l}{N} (n + \frac{1}{2}) \right) \]

Rewrite the cosine terms as the real parts of complex exponentials:
\[ \sum_{n=0}^{N-1} \phi_k[n] = \text{Re} \left( \sum_{n=0}^{N-1} e^{j \frac{\pi}{N} (n+\frac{1}{2})} \right) = \text{Re} \left( e^{j \frac{\pi}{N} \sum_{n=0}^{N-1} e^{j \frac{2\pi l}{N} n}} \right) \]
(continued on next page)
Let

\[ S = \sum_{n=0}^{N-1} e^{j \frac{2\pi l}{N} n} \]

\[ = 1 + e^{j \frac{2\pi l}{N}} + e^{j \frac{2\pi l}{N} 2} + \cdots + e^{j \frac{2\pi l}{N} (N-1)} \]

If \( l = 0 \), then \( S = N \).

Otherwise

\[ S - Se^{j \frac{2\pi l}{N}} = S(1 - e^{j \frac{2\pi l}{N}}) = 1 - e^{j \frac{2\pi l}{N} N} = 1 - e^{j 2\pi l} = 0 \]

and \( S = 0 \).

Thus the sum

\[ \sum_{n=0}^{N-1} \phi_k[n] = N \delta[k] \]
Orthogonality

Show that

$$\sum_{n=0}^{N-1} \phi_k[n] \phi_l[n] = \begin{cases} N & \text{if } k = l = 0 \\ N/2 & \text{if } k = l \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

This orthogonality property is the basis of the analysis equation.
Orthogonality

Show that

$$\sum_{n=0}^{N-1} \phi_k[n] \phi_l[n] = \begin{cases} N & \text{if } k = l = 0 \\ N/2 & \text{if } k = l \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{n=0}^{N-1} \cos \left( \frac{\pi k}{N} \left( n + \frac{1}{2} \right) \right) \cos \left( \frac{\pi l}{N} \left( n + \frac{1}{2} \right) \right) = \frac{1}{2} \sum_{n=0}^{N-1} \cos \left( \frac{\pi (k-l)}{N} \left( n + \frac{1}{2} \right) \right) + \frac{1}{2} \sum_{n=0}^{N-1} \cos \left( \frac{\pi (k+l)}{N} \left( n + \frac{1}{2} \right) \right)$$

Now we can use the previous result to show that the first sum is equal to $\frac{N}{2} \delta[k-l]$ and the second sum is equal to $\frac{N}{2} \delta[k+l]$.

Since both $k$ and $l$ must be between 0 and $N-1$, the first term is $\frac{N}{2}$ if $k = l$ and the second term is $\frac{N}{2}$ if $k = l = 0$. 
Compaction: Gradient

If a signal has predominately low-frequency content, then the higher order coefficients of the DCT tend to decrease faster than the corresponding coefficients of the DFT.

Here are results for a ramp.

Note that the same scales apply for $X_C$ and $X$. 
The same sort of compaction results for sinusoidal signals.

Same scales in each panel.