Discrete-Time Fourier Series

**Synthesis Equation**

\[ x[n] = \sum_{k=\langle N \rangle} X[k] e^{j \frac{2\pi k}{N} n} \]

**Analysis Equation**

\[ X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j \frac{2\pi k}{N} n} \]
Last Time …

Last week we looked at continuous-time Fourier series, as our first exposure to thinking about signals as sums of sinusoids (or equivalently as complex exponentials).

Importance of thinking about signals as sums of sinusoids:

• special \textbf{mathematical} properties of sinusoids

example: orthogonality property

\[
\int_{T} e^{j \frac{2\pi k}{T} t} e^{-j \frac{2\pi l}{T} t} dt = \begin{cases} 
T & \text{if } k = l \\
0 & \text{otherwise}
\end{cases}
\]

This property is the basis of the CTFS analysis equation

• special importance of sinusoids in \textbf{physics} and \textbf{biophysics}

we will touch on that today (sinusoids and hearing)
Today we will look at Discrete-Time Fourier Series

Why discrete?

Early work was in CT (Fourier and his work on heat). As digital electronics became widely available / inexpensive (1960s), it became increasingly desireable to work in discrete domain.

Initially, we thought of DT as an approximation to CT. This perspective is still important. However, it turns out to be simpler and more parsimonious to think of **sampling** and **digital processing** separately.

Following is an example of thinking about a signal as DT.
Discrete-Time Fourier Series

Compare the pitches of two signals:

\[ x_1[n] = \sin(2\pi n/10) \]
\[ x_2[n] = \begin{cases} 
1 & \text{if } \mod(n, 10) < 5 \\
-1 & \text{otherwise}
\end{cases} \]

each played at 4 kHz sample rate.

Different sounds, same pitch. We would like to understand why.
Discrete-Time Fourier Series

Each of the signals in the previous slide is DT. We can think of each cycle as composed of 10 samples. When "played," the DT waveform (say 400 cycles) is converted to CT by the hardware in my laptop. Here is my code:

```python
import matplotlib.pyplot as plt
from wav_utils import *
from math import cos, sin, pi, e
j2pi = complex(0, 2*pi)

# 400 cycles of a unit-amplitude sinusoid
sine = 400*[sin(2*pi*n/10) for n in range(10)]

# 400 cycles of a square wave, amplitude chosen to match loudness
pulse = 400*(5*[2/pi]+5*[-2/pi])

gap = 800*[0]
wav_write(3*(sine+gap+pulse+gap), 4000, 'pitches.wav')
```
Listen to the sounds. Different sounds, same pitch.

Interest in the question of pitch has been longstanding (from music). Musical instruments are typically described as having three properties: loudness, pitch, and timbre (pronounced tam-ber).

**Loudness** is easy to understand intuitively.

**Pitch** has to do with harmoniousness or consonance with other sounds.

**Timbre** is more difficult to define. It’s often defined as the ”quality” of a sound, as distinct from its loudness and pitch.

Why do the sounds on the previous slide have the same pitch? What does that mean physically?
Early pitch experiments were based on stringed instruments and tubes. Both were known to produce not just a fundamental tone (which determines the pitch) but also harmonic overtones.
Pitch Experiments

Early experiments to understand pitch were limited by their ability to manipulate sounds.

For example, musical instruments based on strings or columns of air were known to exhibit a (more or less) characteristic harmonic pattern.

Not clear how to think about harmonics. Do they contribute to pitch? timbre? loudness?

A break through occurred with the work of Seebeck who used sirens to generate more complicated sounds.

Clever experiment ... controversial interpretation.
Seebeck used a siren to generate more complicated sounds (circa 1841). Sounds were made by passing a jet of compressed air through holes in a spinning disk.

The pattern of holes determined the pattern of pulses in each period. The speed of spinning controlled the number of periods per second.
Seebeck found interesting phenomena based on periodicity of holes.

Listen to these signals played with 4 kHz sample rate.
Interpreting Complex Sounds

Code to generate Seebeck’s waveforms.

# Seebeck’s two pulse signals
output = []
for i in range(1,10):
    xi = [1]+9*[0]
    xi[i] = 1
    output += 400*xi
    output += 800*[0]
wav_write(output,4000,’pulses.wav’)
Seebeck interpreted results in terms of the periodicity of the holes (time domain).

Georg Ohm (already known for his work on electrical conduction) interpreted results using Fourier’s recently described decomposition.

A bitter controversy ensued.
Fourier Interpretation

Find Fourier series representations of Seebeck’s signals.
Fourier Interpretation

Find Fourier series representations of Seebeck’s signals.

\[ X_i[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x_i[n] e^{-j\frac{2\pi kn}{N}} n = \frac{1}{10} \sum_{n=0}^{9} x_i[n] e^{-j\frac{2\pi k}{10} n} = \frac{1}{10} \left( 1 + e^{-j\frac{2\pi k}{10} i} \right) \]

DC: \( k = 0 \) term

\[ X_i[0] = \frac{1}{10} \left( 1 + e^{-j\frac{2\pi 0}{10} i} \right) = \frac{2}{10} \]

Fundamental: \( k = 1 \) term

\[ X_i[1] = \frac{1}{10} \left( 1 + e^{-j\frac{2\pi 1}{10} i} \right) \]

\[ |X_i[1]| \]

\[ i \]

\[ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \]
Fourier Matching

Fundamental: \( k = 1 \) term

\[
X_i[1] = \frac{1}{10} \left( 1 + e^{-j\frac{2\pi}{10}i} \right)
\]

<table>
<thead>
<tr>
<th>( X_i[1] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta = \frac{2\pi}{10} )</td>
</tr>
</tbody>
</table>

Which series corresponds to each time function on the left (below)?

\( x_1[n] \)
\( x_2[n] \)
\( x_3[n] \)
\( x_4[n] \)
\( x_5[n] \)
Fourier Matching

Fundamental: \( k = 1 \) term

\[
X_i[1] = \frac{1}{10} \left( 1 + e^{-j\frac{2\pi}{10}i} \right)
\]

| \( X_i[1] \) |

\[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9\]

Which series corresponds to each time function on the left (below)?

\[x_1[n] \quad x_2[n] \quad x_3[n] \quad x_4[n] \quad x_5[n]\]

\[X_1[k] \quad X_2[k] \quad X_3[k] \quad X_4[k] \quad X_5[k]\]

\[\theta = \frac{2\pi}{10}\]
Fourier Series

Why are the magnitude functions for $x_2[\cdot]$ and $x_8[\cdot]$ the same?
Are the series coefficients the same?

\[ x_1[n] \quad \cdots \quad |X_1[k]| \quad \cdots \]
\[ x_2[n] \quad \cdots \quad |X_2[k]| \quad \cdots \]
\[ x_3[n] \quad \cdots \quad |X_3[k]| \quad \cdots \]
\[ x_4[n] \quad \cdots \quad |X_4[k]| \quad \cdots \]
\[ x_5[n] \quad \cdots \quad |X_5[k]| \quad \cdots \]
\[ x_6[n] \quad \cdots \quad |X_6[k]| \quad \cdots \]
\[ x_7[n] \quad \cdots \quad |X_7[k]| \quad \cdots \]
\[ x_8[n] \quad \cdots \quad |X_8[k]| \quad \cdots \]
\[ x_9[n] \quad \cdots \quad |X_9[k]| \quad \cdots \]
Fourier Series

Why are the magnitude functions for $x_2[\cdot]$ and $x_8[\cdot]$ the same? Are the series coefficients the same?

These signals differ by a time-shift

$$x_8[n] = x_2[n - 8]$$

which changes only the phase:

$$X_8[k] = \frac{1}{10} \sum_{n=\langle 10 \rangle} x_8[n] e^{-j \frac{2\pi}{10} n} = \frac{1}{10} \sum_{n=\langle 10 \rangle} x_2[n - 8] e^{-j \frac{2\pi}{10} n}$$

$$= \frac{1}{10} \sum_{m=\langle 10 \rangle} x_2[m] e^{-j \frac{2\pi}{10} (m+8)}$$

$$= \frac{1}{10} e^{-j \frac{2\pi}{10} 8} \sum_{m=\langle 10 \rangle} x_2[m] e^{-j \frac{2\pi}{10} m} = e^{-j \frac{2\pi}{10} 8} X_2[k]$$
Why is the pitch of $x_5[n]$ different from the others?
Missing Fundamental

An important feature of $x_5[\cdot]$ is that it has no fundamental component (when analyzed with $N = 10$). Could this missing fundamental explain the difference in percept?

We can test this hypothesis by removing the fundamental component from each of Seebeck’s waveforms.

How would you create signals $y_i[\cdot]$ that have the same harmonics as $x_i[\cdot]$ but no fundamental?

Write an expression for $y_i[\cdot]$. 
How would you create signals $y_i[\cdot]$ that have the same harmonics as $x_i[\cdot]$ but no fundamental?

Write an expression for $y_i[\cdot]$.

$$y_i[n] = x_i[n] - \frac{1}{10} \left( 1 + e^{-j\frac{2\pi}{10}} \right) e^{j\frac{2\pi}{10} n} - \frac{1}{10} \left( 1 + e^{j\frac{2\pi}{10}} \right) e^{-j\frac{2\pi}{10} n}$$

Listen to these signals: $y_i[\cdot]$ sounds nearly the same as $x_i[\cdot]$!

The reason that $x_5[\cdot]$ sounds different from the other $x_i[\cdot]$ signals is not just because of the missing fundamental!

Link back to beginning: Thinking about these sounds as sums of sinusoids greatly enriched the study of pitch.

Understanding the origin of pitch remains a hotly debated topic. Modern approaches look for neural representations of pitch.
Fourier Series of $X_i[\cdot]$ and $Y_i[\cdot]$

Listen to the signals after the fundamentals have been removed.
Fourier Series of $X_i[.]$ and $Y_i[.]$

Code to generate Seebeck’s signals after fundamental is removed.

```python
# Seebeck’s signals after removing the fundamentals
wave = []
for i in range(1,10):
    xi = [1]+9*[0]
    xi[i] = 1
    yi = [xi[n]-1/10*(1+e**(-j2pi*i/10))*e**(j2pi*n/10)\n          -1/10*(1+e**(j2pi*i/10))*e**(-j2pi*n/10)\n          for n in range(10)]
    yi = [y.real for y in yi]
    wave += 400*yi
    wave += 800*[0]
wav_write(wave,4000,'nofund.wav')
```