Representations of Complex Numbers

Let $c$ represent a complex number.

rectangular form: $c = a + jb$
polar (phasor) form: $r \angle \theta$
Euler form: $r e^{j\theta}$

Find

$$\angle (jc) - \angle (c)$$

which can also be written as

$$\arg (jc) - \arg (c)$$
Representations of Complex Numbers

Find $\angle(jc) - \angle(c)$.

Rectangular coordinates:

$$\angle(c) = \angle(a + jb) = \text{atan2}(b, a)$$

$$\angle(jc) = \angle(ja - b) = \text{atan2}(a, -b)$$

$$\rightarrow \angle(jc) - \angle(c) = \text{atan2}(a, -b) - \text{atan2}(b, a)$$

If you are better at trig than I am, ...

$$\text{atan2}(y_1, x_1) \pm \text{atan2}(y_2, x_2) = \text{atan2}(y_1x_2 \pm y_2x_1, x_1x_2 \mp y_1y_2)$$

$$\text{atan2}(a, -b) - \text{atan2}(b, a) = \text{atan2}(a^2 + b^2, -ab + ba) = \text{atan2}(a^2 + b^2, 0) = \frac{\pi}{2}$$
Representations of Complex Numbers

Find \( \angle(jc) - \angle(c) \).

Graphically:

\[
\begin{align*}
c &= a + jb \\
jc &= ja - b
\end{align*}
\]

From the plot, we see that \( jc \) is a \( \frac{\pi}{2} \) rotation of \( c \).

Therefore \( \angle(jc) - \angle(c) = \frac{\pi}{2} \).
Find $\angle(jc) - \angle(c)$.

Using Euler’s equation:

\[ c = re^{j\theta} \]

\[ jc = jre^{j\theta} = e^{j\frac{\pi}{2}}e^{j\theta} = e^{j(\theta + \frac{\pi}{2})} \]

Therefore $\angle(jc) - \angle(c) = \theta + \frac{\pi}{2} - \theta = \frac{\pi}{2}$. 
Complex Numbers

How many of the following are true?

- \(\frac{1}{\cos \theta + j \sin \theta} = \cos \theta - j \sin \theta\)
- \((\cos \theta + j \sin \theta)^n = \cos(n\theta) + j \sin(n\theta)\)
- \(|2 + j2 + e^{\frac{j\pi}{4}}| = |2 + j2| + |e^{\frac{j\pi}{4}}|\)
- \(\text{Im}(j^j) > \text{Re}(j^j)\)
- \(\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}1\)
Complex Numbers

\[
\frac{1}{\cos \theta + j \sin \theta} = \frac{?}{\cos \theta - j \sin \theta}
\]

\[
\cos \theta + j \sin \theta = e^{j\theta}
\]

\[
\frac{1}{\cos \theta + j \sin \theta} = \frac{1}{e^{j\theta}} = e^{-j\theta} = \cos(-\theta) + j \sin(-\theta) = \cos \theta - j \sin \theta
\]

\[
\frac{1}{\cos \theta + j \sin \theta} = \cos \theta - j \sin \theta \quad \checkmark
\]
Complex Numbers

\[(\cos \theta + j \sin \theta)^n \quad ? \quad \cos(n\theta) + j \sin(n\theta)\]

\[(\cos \theta + j \sin \theta)^n = (e^{j\theta})^n = e^{jn\theta} = \cos(n\theta) + j \sin(n\theta)\]

\[(\cos \theta + j \sin \theta)^n = \cos(n\theta) + j \sin(n\theta) \quad \checkmark\]
Complex Numbers

\[ |2 + j2 + e^{\frac{j\pi}{4}}| \quad ? \quad |2 + j2| + |e^{\frac{j\pi}{4}}| \]

\[ |2 + j2 + e^{\frac{j\pi}{4}}| = |2\sqrt{2} e^{\frac{j\pi}{4}} + e^{\frac{j\pi}{4}}| \]
\[ = |(2\sqrt{2} + 1)e^{\frac{j\pi}{4}}| \]
\[ = |(2\sqrt{2} + 1)||e^{\frac{j\pi}{4}}| \]
\[ = 2\sqrt{2} + 1 \]

\[ |2 + j2| + |e^{\frac{j\pi}{4}}| = 2\sqrt{2} + 1 \]

\[ |2 + j2 + e^{\frac{j\pi}{4}}| = |2 + j2| + |e^{\frac{j\pi}{4}}| \quad \checkmark \]

This is only true because the angles of \(2 + j2\) and \(e^{\frac{j\pi}{4}}\) are equal!

\(|a + b|\) is NOT generally equal to \(|a| + |b|\).
Complex Numbers

\[ \text{Im}(j^j) > \text{Re}(j^j) \]

\[ j^j = \left(e^{j\pi/2}\right)^j = e^{-\pi/2} \] which is real and \( > 0 \).

Therefore \( \text{Im}(j^j) = 0 \) and is always less than the real part.

Caveat: There are other ways to express \( j \).

\[ j^j = (e^{j2\pi(n+\frac{1}{4})})^j = e^{-2\pi(n+\frac{1}{4})} \]

All of these alternatives lead to real numbers that are \( > 0 \).
Therefore the original premise is always false.

\[ \text{Im}(j^j) > \text{Re}(j^j) \quad \times \]

Notice that \( j^j \) is multi-valued, much like the \( n^{\text{th}} \) root of 1.
Complex Numbers

\[ \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}1 \]

Let \( c_1 = 2+j \) and \( c_2 = 3+j \) so that \( c_3 = (2+j)(3+j) = 5+5j \).

The angle of a product is the sum of the angles of the constituents:

\[ \angle c_1 + \angle c_2 = \angle c_3 \]

This proves the premise.

More generally,

\( c_1 \) could be any complex number whose angle is \( \tan^{-1}\left(\frac{1}{2}\right) \),
\( c_2 \) could be any complex number whose angle is \( \tan^{-1}\left(\frac{1}{3}\right) \),
and the product \( c_1c_2 \) would have angle \( \tan^{-1}(1) \),

\[ \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}1 \quad \checkmark \]
Complex Numbers

How many of the following are true?

• \( \frac{1}{\cos \theta + j \sin \theta} = \cos \theta - j \sin \theta \) \[ \checkmark \]

• \((\cos \theta + j \sin \theta)^n = \cos n\theta + j \sin n\theta \) \[ \checkmark \]

• \(|2 + j2 + e^{j\frac{\pi}{4}}| = |2 + j2| + |e^{j\frac{\pi}{4}}| \) \[ \checkmark \]

• \(\text{Im}(j^j) > \text{Re}(j^j) \) \[ \times \]

• \(\tan^{-1} \left( \frac{1}{2} \right) + \tan^{-1} \left( \frac{1}{3} \right) = \tan^{-1} 1 \) \[ \checkmark \]
Squares

Squares that are $1 \times 1$, $a \times a$, $a^2 \times a^2$, etc., are arranged side-by-side as shown below.
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The upper left corners of these squares can be connected with a straight line, shown in blue.
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![Diagram of squares](image)

The upper left corners of these squares can be connected with a straight line, shown in blue. Determine the slope of the blue line.
Squares

Squares with areas $1 \times 1$, $a \times a$, $a^2 \times a^2$, etc., are arranged side-by-side as shown below.

The upper left corners of these squares can be connected with a straight line, shown in blue. Determine the slope of the blue line.

$$\text{slope} = - \frac{1 - a}{1}$$
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$$\text{slope} = -\frac{1 - a}{1} = -\frac{1}{\sum a^n}$$
Squares

Squares with areas $1 \times 1$, $a \times a$, $a^2 \times a^2$, etc., are arranged side-by-side as shown below.

The upper left corners of these squares can be connected with a straight line, shown in blue. Determine the slope of the blue line.

The slope is given by:

$$\text{slope} = -\frac{1 - a}{1} = -\frac{1}{\sum a^n}$$

Therefore

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1 - a}$$
Mathematically

Determine the sum of a geometric sequence. Let

\[ S = \sum_{n=0}^{\infty} a^n = 1 + a + a^2 + a^3 + \cdots \]

Then

\[ aS = a + a^2 + a^3 + a^4 + \cdots \]

so that

\[ S - aS = (1 - a)S = 1 \]

and

\[ S = \frac{1}{1 - a} \]

provided \(|a| < 1\).
Computationally

How fast does the sum converge?

Compute partial sums

\[ S_N = \sum_{n=0}^{N-1} a^n \]

What is the shape of \( S_N(N) \)?

How does the shape change as a function of \( a \)?

Try \( a = \frac{1}{2} \) and \( \frac{99}{100} \).
Power Square

Find the sum of the numbers in the infinite quadrant shown below, where \( a < 1 \).

\[ \begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & \\
4 & 5 & 6 & 7 & 8 & \ldots \\
3 & 4 & 5 & 6 & 7 & \ldots \\
2 & 3 & 4 & 5 & 6 & \ldots \\
1 & 2 & 3 & 4 & 5 & \ldots \\
0 & 1 & 2 & 3 & 4 & \ldots \\
\end{array} \]
Find the sum of the numbers in the infinite quadrant shown below, where $a < 1$.

\[
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\vdots & a^2 & a^3 & a^4 & a^5 & \\
a & a^3 & a^4 & a^5 & a^6 & \\
a^2 & a^3 & a^4 & a^5 & a^6 & \\
a^3 & a^4 & a^5 & a^6 & a^7 & \\
a^4 & a^5 & a^6 & a^7 & a^8 & \\
a^5 & a^6 & a^7 & a^8 & a^9 & \\
\end{array}
\]

\[
\text{sum} = 1a^0 + 2a^1 + 3a^2 + \cdots = \sum_{n=1}^{\infty} na^{n-1}
\]
Power Square

Find the sum of the numbers in the infinite quadrant shown below, where \( a < 1 \).

\[
\begin{align*}
& a^4 \quad a^5 \quad a^6 \quad a^7 \quad a^8 \quad \cdots \quad = \quad a^4 \times \frac{1}{1-a} \\
& a^3 \quad a^4 \quad a^5 \quad a^6 \quad a^7 \quad \cdots \quad = \quad a^3 \times \frac{1}{1-a} \\
& a^2 \quad a^3 \quad a^4 \quad a^5 \quad a^6 \quad \cdots \quad = \quad a^2 \times \frac{1}{1-a} \\
& a^1 \quad a^2 \quad a^3 \quad a^4 \quad a^5 \quad \cdots \quad = \quad a \times \frac{1}{1-a} \\
& a^0 \quad a^1 \quad a^2 \quad a^3 \quad a^4 \quad \cdots \quad = \quad \frac{1}{1-a} \\
& \text{sum} = 1a^0 + 2a^1 + 3a^2 + \cdots = \sum_{n=1}^{\infty} na^{n-1} = \left( \frac{1}{1-a} \right)^2
\end{align*}
\]
Mathematically

Determine a closed-form expression for the following sum:

$$\sum_{n=0}^{\infty} na^{n-1}$$

Let

$$S = \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

Differentiating both sides by \(a\) yields

$$\frac{d}{da} \sum_{n=0}^{\infty} a^n = \sum_{n=0}^{\infty} \frac{d}{da} a^n = \sum_{n=0}^{\infty} na^{n-1}$$

$$= \frac{d}{da} \left( \frac{1}{1-a} \right) = \left( \frac{1}{1-a} \right)^2$$

Thus

$$\sum_{n=0}^{\infty} na^{n-1} = \left( \frac{1}{1-a} \right)^2$$
Each diagram below shows the unit circle in the complex plane, with the origin labeled with a dot. The lines represent the sum

\[ S = \sum_{n=0}^{100} \alpha^n. \]

Determine the diagram for which \( \alpha = 0.8 + 0.2j. \)
The first two terms in the series representation of $S$ are 1 and $0.8 + 0.2j$. All of the curves start with a horizontal line to the right that stops at the intersection with the unit circle. Thus all of the curves correctly represent the first term. The second term should have a length that is somewhat shorter than 1 and an angle of $\tan^{-1}(0.25)$. We can immediately discard curves C, D, and E because their second terms have negative imaginary parts, which is wrong.

How quickly should the terms in the series converge? The magnitude of $\alpha$ is $\sqrt{0.8^2 + 0.2^2} \approx 0.8$. The magnitude of the last term in the sum is then approximately $0.8^{100} \approx \frac{1}{2}^{33} \approx \frac{1}{1000^3} < 10^{-9}$ — which would not be visible in the plots. Thus, the curve should appear to converge to a limit, which eliminates curves A and H. Furthermore, it says that the sum of 100 terms is very nearly equal to the infinite sum:

$$S \approx S_\infty = \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1 - \alpha} = \frac{1}{1 - (0.8 + 0.2j)} = \frac{1}{0.2 - 0.2j}$$

$$= \frac{1}{0.2\sqrt{2}e^{-j\pi/4}} = \frac{5}{\sqrt{2}}e^{j\pi/4}$$

Thus the final values of B and G are wrong, and the answer is F.
Trig Table

\[ \sin(a+b) = \sin(a) \cos(b) + \cos(a) \sin(b) \]
\[ \sin(a-b) = \sin(a) \cos(b) - \cos(a) \sin(b) \]
\[ \cos(a+b) = \cos(a) \cos(b) - \sin(a) \sin(b) \]
\[ \cos(a-b) = \cos(a) \cos(b) + \sin(a) \sin(b) \]
\[ \tan(a+b) = \frac{\tan(a)+\tan(b)}{1-\tan(a) \tan(b)} \]
\[ \tan(a-b) = \frac{\tan(a)-\tan(b)}{1+\tan(a) \tan(b)} \]
\[ \sin(A) + \sin(B) = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) \]
\[ \sin(A) - \sin(B) = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right) \]
\[ \cos(A) + \cos(B) = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) \]
\[ \cos(A) - \cos(B) = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right) \]

\[ \sin(a+b) + \sin(a-b) = 2 \sin(a) \cos(b) \]
\[ \sin(a+b) - \sin(a-b) = 2 \cos(a) \sin(b) \]
\[ \cos(a+b) + \cos(a-b) = 2 \cos(a) \cos(b) \]
\[ \cos(a+b) - \cos(a-b) = -2 \sin(a) \sin(b) \]

\[ 2 \cos(A) \cos(B) = \cos(A-B) + \cos(A+B) \]
\[ 2 \sin(A) \sin(B) = \cos(A-B) - \cos(A+B) \]
\[ 2 \sin(A) \cos(B) = \sin(A+B) + \sin(A-B) \]
\[ 2 \cos(A) \sin(B) = \sin(A+B) - \sin(A-B) \]