You have three hours.
This quiz is closed book, but you may use three 8.5 \times 11 sheets of paper (six sides). No calculators, computers, cell phones, music players, or other electronic devices.
1. Discrete-Time Transforms \([24\ \text{points}]\)

Part 1.

Let \(X_1(\Omega)\) represent the Discrete-Time Fourier Transform (DTFT) of the signal

\[
x_1[n] = \begin{cases} 
-2 & n = -2 \\
1 & n = -1, 0, 1 \\
2 & n = 2 \\
0 & \text{otherwise}
\end{cases}
\]

Determine a closed form expression for \(X_1(\Omega)\) in terms of \(\sin\) and \(\cos\) functions (no complex exponentials), and enter your expression in the box below.

\[
1 + 2 \cos(\Omega) - 4j \sin(2\Omega)
\]

Let \(X_2(\Omega)\) represent the DTFT of the signal

\[
x_2[n] = (-j)^n + e^{j\pi n/4}
\]

Write a closed form expression for \(X_2(\Omega)\) in the box below.

\[
2\pi\delta\left(\Omega + \frac{\pi}{2}\right) + 2\pi\delta\left(\Omega - \frac{\pi}{4}\right)
\]
Part 2.

Let $X_3[k]$ represent the Discrete Fourier Transform (DFT) of $x_3[n]$ computed with $N = 6$. Define a new signal $y_3[n]$ in terms of $x_3[n]$:

$$y_3[n] = 9 - 2x_3[n]$$

Let $Y_3[k]$ represent the DFT of $y_3[n]$. Determine expressions for $Y_3[k]$ in terms of $X_3[k]$, and enter those expressions in the boxes below.

\[
\begin{align*}
y_3[0] &= 9 - 2X_3[0] \\
\end{align*}
\]

Let $X_4[k]$ represent the DFT of $x_4[n]$ computed with $N = 6$. Define a new signal

$$y_4[n] = 5(-1)^n x_4[n]$$

Let $Y_4[k]$ represent the DFT of $y_4[n]$. Determine expressions for $Y_4[k]$ in terms of $X_4[k]$, and enter those expressions in the boxes below.

\[
\begin{align*}
y_4[0] &= 5X_4[-3] \\
y_4[1] &= 5X_4[-2] \\
y_4[2] &= 5X_4[-1] \\
y_4[3] &= 5X_4[0] \\
\end{align*}
\]
**Part 3.**

Let $F_i[k_r, k_c]$ represent the Discrete Fourier Transforms (DFTs) of the following two-dimensional signals $f_i[r, c]$, where $0 \leq r < 64$ and $0 \leq c < 64$. For each signal, list all of the indices of the form $[k_r, k_c]$, where $-31 \leq k_r \leq 32$ and $-31 \leq k_c \leq 32$, for which $F_i[k_r, k_c] \neq 0$ in the box provided.

Notice that $f_i[r, c]$ may have positive, negative, or even complex values.

<table>
<thead>
<tr>
<th>Expression</th>
<th>List of 2D Indices ($[k_r, k_c]$) for which $F_i[k_r, k_c] \neq 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\cos \frac{6\pi r}{64}) (\cos \frac{8\pi c}{64})$</td>
<td>[3, 4], [3, -4], [-3, 4], [-3, -4]</td>
</tr>
<tr>
<td>$(\cos \frac{6\pi r}{64}) + (\cos \frac{8\pi c}{64})$</td>
<td>[3, 0], [-3, 0], [0, 4], [0, -4]</td>
</tr>
<tr>
<td>$\cos (\frac{6\pi r}{64} + \frac{8\pi c}{64})$</td>
<td>[3, 4], [-3, -4]</td>
</tr>
<tr>
<td>$\cos (\frac{6\pi r}{64} - \frac{8\pi c}{64})$</td>
<td>[3, -4], [-3, 4]</td>
</tr>
<tr>
<td>$\sin (\frac{6\pi r}{64} + \frac{8\pi c}{64})$</td>
<td>[3, 4], [-3, -4]</td>
</tr>
<tr>
<td>$(1 + \cos \frac{6\pi r}{64}) (1 + \cos \frac{8\pi c}{64})$</td>
<td>[0, 0], [3, 0], [-3, 0], [0, 4], [0, -4], [3, 4], [3, -4], [-3, 4], [-3, -4]</td>
</tr>
<tr>
<td>$e^{j2\pi \frac{3\pi r}{64}} e^{j2\pi \frac{4\pi c}{64}}$</td>
<td>[3, 4]</td>
</tr>
<tr>
<td>$e^{j2\pi \frac{3\pi r}{64}} + e^{j2\pi \frac{4\pi c}{64}}$</td>
<td>[3, 0], [0, 4]</td>
</tr>
<tr>
<td>$(-1)^r + (-1)^c$</td>
<td>[32, 0], [0, 32]</td>
</tr>
<tr>
<td>$(\cos \frac{32\pi r}{64})^2$</td>
<td>[0, 0], [32, 0]</td>
</tr>
</tbody>
</table>
Worksheet (intentionally blank)
2. Surveying  

[20 points]

Farmland in the midwestern United States is often divided into "sections" of uniform size. When viewed from above, this land often looks much like a grid, as can be seen in the photograph below:

![Farmland grid](image)

The following shows the magnitudes of a portion of the DFT coefficients of a different satellite photograph of farmland, taken from directly overhead.

![DFT coefficients](image)

In order to answer the questions on the following page, you may assume that the sections in the image are nearly identical, nearly rectangular (but not necessarily square), and laid out on a nearly perfect grid. Given this information, answer the questions on the following page about the image whose DFT coefficients are shown above.
Approximately how many sections of land are there in the image whose DFT coefficients are shown on the previous page? \[ \approx 112 \]

Explain your method.

We can view this DFT as something like the sum of two DFTs, where one is given by, approximately:

\[
F_1[k_x, k_y] = \sum_{m=-\infty}^{\infty} \delta[k_x - 14m] \delta[k_y]
\]

and the other is given by:

\[
F_2[k_x, k_y] = \sum_{m=-\infty}^{\infty} \delta[k_y - 8m] \delta[k_x]
\]

The first tells us we have something that is repeating around 14 times over the course of the image horizontally, and the second tells us that we have something that is repeating around 8 times vertically. Given this, we can infer that there are around 14 sections horizontally, and around 8 sections vertically. In total, this gives us around 112 sections.

If the image represented an area that was 10.5 miles wide and 4.0 miles tall, what are the approximate dimensions of each of the sections of land, in miles?

Height: \[ \approx 0.5 \text{ mi} \]

Width: \[ \approx 0.75 \text{ mi} \]

Explain your method.

Using the intermediate results from above, the width of a section is \( \frac{10.5 \text{ mi}}{14 \text{ sections}} = 0.75 \frac{\text{mi}}{\text{section}} \).

Similarly, the height of a section is around \( \frac{4 \text{ mi}}{8 \text{ sections}} = 0.5 \frac{\text{mi}}{\text{section}} \).
3. Filtering \([22 \text{ points}]\)

**Part 1.**

For this problem, we will start with a DT filter whose frequency response is:

\[
H(\Omega) = H(\Omega + 2\pi) = \begin{cases} 
1, & \text{if } 0 \leq |\Omega| < \frac{\pi}{3} \\
0, & \text{otherwise}
\end{cases}
\]

Now, consider constructing a new filter of the following form based on \(H:\)

\[
x[n] \rightarrow w[n] \rightarrow H(\Omega) \rightarrow z[n] \rightarrow y[n]
\]

\((-1)^{n}\)

Mathematically, the relationship between the signals \(x[n]\) and \(y[n]\) is given by:

\[
w[n] = (-1)^{n} \times x[n] \\
z[n] = (w * h)[n] \\
y[n] = (-1)^{n} \times z[n]
\]

Consider applying this new filter to a signal \(x_1\), whose DTFT magnitudes are shown below:

On the following page, sketch a graph of the DTFT magnitudes of the resulting signal. Label all key points on your graph.
Part 2.

Consider the following image, which we'll refer to as $x[r,c]$. This image has height $R$ and width $C$.

In addition, consider several other signals:

$$H_1[k_r,k_c] = j \sin \left( \frac{10\pi k_r}{R} \right)$$

$$h_2[r,c] = \sin \left( \frac{10\pi r}{R} \right)$$

For each of the expressions below, indicate which of the images on the following page is represented by that expression. In each of those images, black represents the lowest value (not necessarily 0), and white represents the highest value (not necessarily 1).

$$(x \times h_1)[r,c]$$

$$(x \times h_2)[r,c]$$

$$(x \ast h_1)[r,c]$$

$$(x \ast h_2)[r,c]$$
4. Time and Frequency Patterns \[16 \text{ points}\]

Eight time waveforms are shown in the left panels below. The corresponding Fourier transform magnitudes are shown in the right panels, however, the order has been shuffled. For each panel on the right, find the corresponding time waveform from the left panels, and enter its number in the box provided.

Each time function is plotted on the same time scale, and each frequency function is plotted on the same frequency scale. Time functions are zero outside the range shown.
The time waveforms can be classified as having three important parameters:

- **period**: The periods of $x_1$, $x_3$, $x_5$, and $x_7$ are half as long as those of the others.

- **shape**: The overall shape of the waveform is either triangular or square, and can be thought of as multiplying an underlying periodic time signal.

- **overall length**: $x_1$, $x_2$, $x_3$, and $x_4$ are short, the others are long.

These parameters affect the magnitudes of the Fourier transforms in distinct ways. Let A-H represent the waveforms in the right column.

- The period in time is inversely related to the period in frequency. Thus A, B, D, and F (which have longer periods in frequency) correspond to $x_1$, $x_3$, $x_5$, and $x_7$.

- Since the shape multiplies the time waveform, it convolves with the frequency waveform. The shape is long compared to the periodicity of the time waveform, therefore the convolution affects the short timescale (i.e., between the periods). The square (in time) has more high frequencies than the triangle, so the square in frequency has larger overshoot. Thus A, E, F, and H correspond to squares ($x_3$, $x_4$, $x_7$, and $x_8$), and the others correspond to triangles.

- The overall length is long compared to the periodicity of the time waveform, therefore the convolution affects the short timescale (i.e., between the periods). The longer the shape, the shorter the spread around each lobe in the frequency domain. Therefore, the broad lobes (A, D, G, and H) correspond to the short overall lengths ($x_1$, $x_2$, $x_3$, and $x_4$).

The answer provided on the previous page is the only combination that satisfies all three of these constraints.
5. **Echos**  

Assume that a single echo interferes with a speaker’s voice that is being recorded by a microphone as illustrated in the following figure.

![Diagram](https://via.placeholder.com/150)

We can represent this recording situation as a linear, time-invariant system, with the speaker’s voice as the input and the recorded microphone signal as the output. Assume that the impulse response of this system is

\[ h(t) = \delta(t-T_1) + \epsilon \delta(t-T_2) \]

where \( T_1 \) represents that delay of the direct path from speaker to microphone, \( T_2 \) represents that delay through the echo path, and \( \epsilon \) represents the amplitude of the echo.
Part 1.

On the axes below, sketch the magnitude and angle of the frequency response of this system in the absence of an echo (i.e., when $\epsilon = 0$). Label all important values.

Here, we are interested in a signal $\delta(t - T_1)$. The CTFT of this signal is $e^{-j\omega T_1}$, which has a constant magnitude of 1 and an angle that varies linearly with $\omega$. 

\[ |H(\omega)| \]

\[ \angle H(\omega) \ [\text{rad}] \]

slope: $-T_1$
Part 2.

Now we’ll consider the case where there is an echo. The following plots show that magnitude and angle of the frequency response of this system for $|\omega| < 1500 \text{ rad/s}$, for some values of $T_1$, $T_2$, and $\epsilon$.

Determine values of $T_1$, $T_2$, and $\epsilon$ that are consistent with the graphs above.

\[
T_1 = \frac{\pi}{1500}
\]

\[
T_2 = \frac{\pi}{300}
\]

\[
\epsilon = 0.2
\]
Take the Fourier transform of \( h(t) \) to obtain the frequency response

\[
H(j\omega) = e^{-j\omega T_1} + \epsilon e^{-j\omega T_2} = e^{-j\omega T_1} \left(1 + \epsilon e^{-j\omega(T_2-T_1)}\right).
\]

The magnitude function

\[
|H(j\omega)| = \left| e^{-j\omega T_1} \right| \left| \left(1 + \epsilon e^{-j\omega(T_2-T_1)}\right) \right|
= \sqrt{1 + 2\epsilon \cos \omega(T_2 - T_1) + \epsilon^2}
\]

oscillates between \( 1+\epsilon \) and \( 1-\epsilon \) with a period (in \( \omega \)) of \( 2\pi/(T_2-T_1) \). From the magnitude plot on the previous page, we can see that \( \epsilon \approx 0.2 \) and \( 2\pi/(T_2 - T_1) \approx 1500/2 \), so that \( T_2 - T_1 \approx \frac{4\pi}{1500} \).

The angle function is

\[
\angle H(j\omega) = -\omega T_1 + \angle \left(1 + \epsilon e^{-j\omega(T_2-T_1)}\right)
\]

Since \( \epsilon \) is small compared to 1, the first term dominates the second, which oscillates about an average value near 0. Thus we can estimate \( T_1 \) from the average slope of the angle plot on the previous page,

\[
T_1 = \frac{\pi}{1500}.
\]

Then \( T_2 \approx \frac{4\pi}{1500} + \frac{\pi}{1500} = \frac{\pi}{300} \).
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