6.003
Signal Processing

Week 4, Lecture B:

Fourier Transform Properties,
Duality

Adam Hartz
hz@mit.edu

Quiz 1

Thursday, 7 March, 2:05-3:55pm, 50-340 (Walker Gym).

No lecture on 7 March or 5 March.
5 March: extra office hours 2-5pm.

The quiz is closed book.
No electronic devices (including calculators).

You may use one 8.5x11" sheet of notes (handwritten or printed, front and back).

Coverage: lectures, recitations, homeworks, and labs up to and including March 5.

Practice materials have been posted to the web.

Last Time: Fourier Transform

Last time, we extended Fourier analysis to aperiodic signals by introducing CTFT and DTFT:

CTFT

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

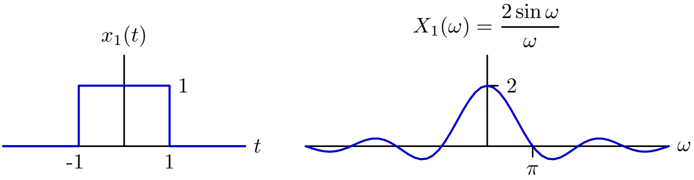
$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

DTFT

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

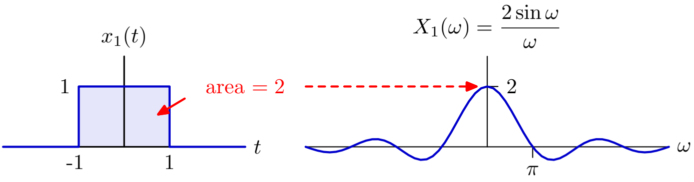
Last Time: FT of Square Pulse



$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-1}^{1} e^{-j\omega t}dt = \frac{2\sin(\omega)}{\omega}$$

Square Pulse

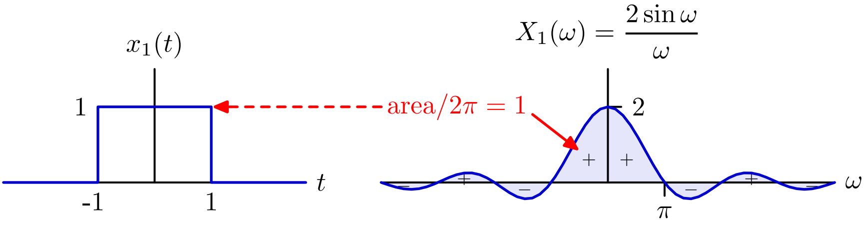
The value of $X(0)$ is the integral of $x(t)$ over all time:



$$X(0) = \int_{-\infty}^{\infty} x(t)e^{-j0t}dt = \int_{-1}^{1} dt = 2$$

Square Pulse

The value of $x(0)$ is the integral of $X(\omega)$ over all frequencies, divided by 2π:

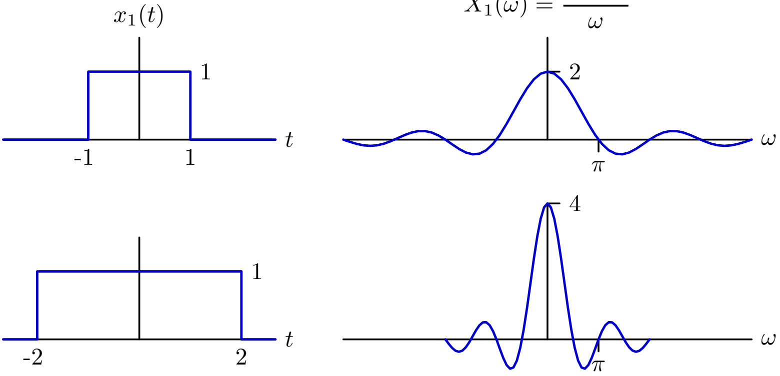


$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega 0}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)d\omega$$

Stretching Time

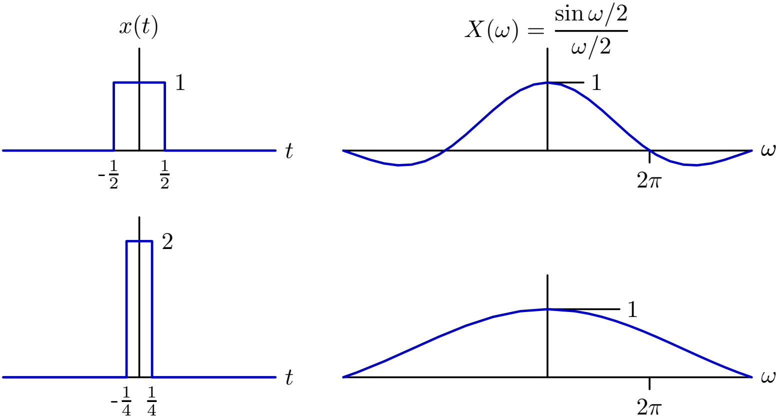
Stretching time compresses frequency and increases amplitude (preserving area).

$$X_1(\omega) = \frac{2\sin\omega}{\omega}$$



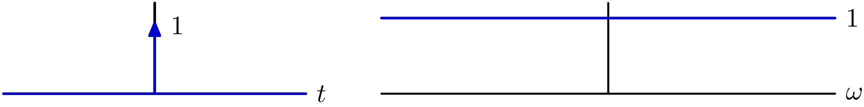
Compressing Time to the Limit

Alternatively, compress time while keeping area = 1:



$\delta(\cdot)$

In the limit, the pulse has zero width but still area 1!
We represent this limit with the delta function: $\delta(\cdot)$.



$\delta(\cdot)$ only has nonzero area, but it has finite area: it is most easily described via an integral:

$$\int_{-\infty}^{\infty} \delta(t)dt = \int_{0_-}^{0_+} \delta(t)dt = 1$$

Importantly, it has the following property (the "sifting property"):

$$\int_{-\infty}^{\infty} \delta(t-a)f(t)dt = f(a)$$

Example: FT of complex exponential

Consider finding the FT of

$$x(t) = e^{j4t}$$

We need $X(\omega)$ such that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega = e^{j4t}$$

Using the sifting property of the delta function, we find:

$$X(\omega) = 2\pi\delta(\omega - 4)$$

Check Yourself!

What is the FT of the following function?

$$x(t) = 2\cos(t)$$

DT impulse: $\delta[\cdot]$

DT equivalent is

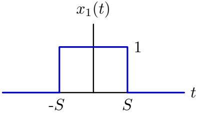
$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

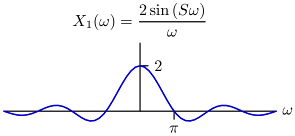
which still has the "sifting property:"

$$\sum_{n=-\infty}^{\infty} \delta[n-a] f[n] = f[a]$$

Check Yourself!

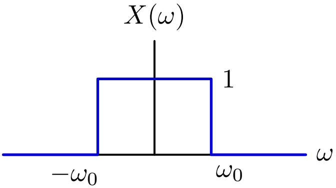
The FT of a square pulse is a "sinc" function:





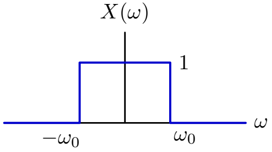
$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-S}^{S} e^{-j\omega t}dt = \frac{2\sin(S\omega)}{\omega}$$

Find the time function whose Fourier transform is:

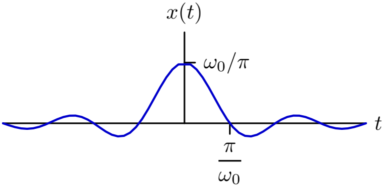


Check Yourself!

Find the time function whose Fourier transform is:



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega t}d\omega - = \frac{\sin(\omega_0 t)}{\pi t}$$



Duality

The Fourier transform and its inverse are symmetric!

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

except for the minus sign in the exponential, and the 2π factor.

Duality

The Fourier transform and its inverse are symmetric!

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

except for the minus sign in the exponential, and the 2π factor.

So, in general, we can say that:

If $x(t)$ has Fourier transform $X(\omega)$, then
$X(t)$ has Fourier transform $2\pi x(-\omega)$.

Duality

Verifying duality:

$$2\pi x(-\omega) \stackrel{?}{=} \mathrm{FT}\left\{X(t)\right\}$$

$$2\pi x(-\omega) \stackrel{?}{=} \int X(t)e^{-j\omega t}dt$$

$$x(-\omega) \stackrel{?}{=} \frac{1}{2\pi}\int X(t)e^{-j\omega t}dt$$

$$x(\omega) \stackrel{?}{=} \frac{1}{2\pi}\int X(t)e^{-j(-\omega)t}dt$$

$$x(t) = \frac{1}{2\pi}\int X(\omega)e^{j\omega t}d\omega$$

Duality

Consequence of duality: "transform pairs"

FT Properties

Duality can often be used to simplify analysis.

Additionally, we can show that certain time-domain operations have equivalent operations in the frequency domain, which can aid the analysis of new signals.

CTFT Properties: Linearity

Consider

$$x(t) = Ax_1(t) + Bx_2(t)$$

Then:

$$\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} \left(Ax_1(t) + Bx_2(t)\right) e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} \left(Ax_1(t)e^{-j\omega t} + Bx_2(t)e^{-j\omega t}\right) dt \\
&= \int_{-\infty}^{\infty} Ax_1(t)e^{-j\omega t}dt + \int_{-\infty}^{\infty} Bx_2(t)e^{-j\omega t}dt \\
&= AX_1(\omega) + BX_2(\omega)
\end{aligned}$$

CTFT Properties: Time Reversal

Consider

$$y(t) = x(-t)$$

Then:

$$Y(\omega) = \int_{-\infty}^{\infty} x(-t)e^{-j\omega t}dt$$

substitute $u = -t$

$$= \int_{\infty}^{-\infty} x(u)e^{-j\omega(-u)}(-du)$$

$$= \int_{-\infty}^{\infty} x(u)e^{-j\omega(-u)}du$$

$$= \int_{-\infty}^{\infty} x(u)e^{-j(-\omega)u}du$$

$$= X(-\omega)$$

CTFT Properties: Time Delay

Consider

$$y(t) = x(t - t_0)$$

Then:

$$Y(\omega) = \int_{-\infty}^{\infty} x(t - t_0) e^{-j\omega t} dt$$

substitute $u = t - t_0$

$$= \int_{-\infty}^{\infty} x(u) e^{-j\omega(u+t_0)} du$$

$$= \int_{-\infty}^{\infty} x(u) e^{-j\omega u} e^{-j\omega t_0} du$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(u) e^{-j\omega u} du$$

$$= e^{-j\omega t_0} X(\omega)$$

CTFT Properties: Conjugation

Consider

$$y(t) = x^*(t)$$

Then:

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt$$

$$Y(\omega) = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt$$

$$Y(\omega) = \int_{-\infty}^{\infty} x^*(t)e^{j(-\omega)t}dt$$

$$Y(\omega) = X^*(-\omega)$$

CTFT Properties: Scaling Time

Consider

$$y(t) = x(At)$$

with $A > 0$ (not flipping in time)

Then:

$$Y(\omega) = \int_{-\infty}^{\infty} x(At) e^{-j\omega t} dt$$

substitute $u = At$

$$= \int_{-\infty}^{\infty} x(u) e^{-j\omega \frac{u}{A}} \frac{du}{A}$$

$$= \int_{-\infty}^{\infty} x(u) e^{-j\frac{\omega}{A}u} \frac{du}{A}$$

$$= \frac{1}{A} \int_{-\infty}^{\infty} x(u) e^{-j\frac{\omega}{A}u} du$$

$$= \frac{1}{A} X\left(\frac{\omega}{A}\right)$$

CTFT Properties: Time Derivative

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

differentiate both sides w.r.t t

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\left(j\omega e^{j\omega t}\right)d\omega$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(j\omega X(\omega)\right)e^{j\omega t}d\omega$$

This shows that the Fourier transform of $\frac{\mathrm{d}x(t)}{\mathrm{d}t}$ is $j\omega X(\Omega)$
Consequently, the FT of $\int x(t)dt$ is $\frac{1}{j\omega}X(\omega)$

CTFT Properties: Frequency Derivative

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

differentiate both sides w.r.t ω

$$\frac{\mathrm{d}X(\omega)}{\mathrm{d}\omega} = \int_{-\infty}^{\infty} x(t) \left(-jte^{-j\omega t}\right) dt$$

$$\frac{\mathrm{d}X(\omega)}{\mathrm{d}\omega} = \int_{-\infty}^{\infty} \left(-jtx(t)\right) e^{-j\omega t} dt$$

multiply both sides by j

$$j\frac{\mathrm{d}X(\omega)}{\mathrm{d}\omega} = \int_{-\infty}^{\infty} \left(tx(t)\right) e^{-j\omega t} dt$$

This shows that the Fourier transform of $tx(t)$ is $j\frac{\mathrm{d}X(\omega)}{\mathrm{d}\omega}$

Table of CTFT Properties

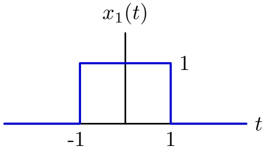
Property	$y(t)$	$Y(\omega)$
Linearity	$Ax_1(t) + Bx_2(t)$	$AX_1(\omega) + BX_2(\omega)$
Time reversal	$x(-t)$	$X(-\omega)$
Time Delay	$x(t - t_0)$	$e^{-j\omega t_0}X(\omega)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Scaling Time	$x(At)$	$\frac{1}{\|A\|}X\left(\frac{\omega}{A}\right)$
Time Derivative	$\frac{\mathrm{d}x(t)}{\mathrm{d}t}$	$j\omega X(\omega)$
Frequency Derivative	$tx(t)$	$j\frac{\mathrm{d}}{\mathrm{d}\omega}X(\omega)$

Table of DTFT Properties

Property	$y[n]$	$Y(\Omega)$
Linearity	$Ax_1[n] + Bx_2[n]$	$AX_1(\Omega) + BX_2(\Omega)$
Time reversal	$x(-n)$	$X(-\Omega)$
Time Delay	$x(n - n_0)$	$e^{-j\Omega n_0}X(\Omega)$
Conjugation	$x^*[n]$	$X^*(-\Omega)$
Frequency Derivative	$nx[n]$	$j\frac{\mathrm{d}}{\mathrm{d}\Omega}X(\Omega)$

Example: Using Properties

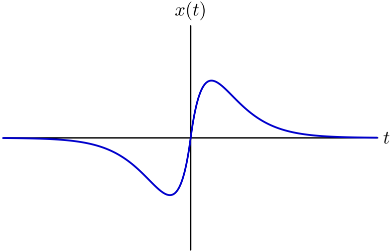
Consider again finding the FT of the function shown below:



Using properties can simplify the analysis!

Example: Using Properties

Consider finding the Fourier transform of $x(t) = 2te^{-3|t|}$, shown below:



Using properties can simplify the analysis!