6.003 Signal Processing

Week 3, Lecture B:
DT Fourier Series

Adam Hartz
hz@mit.edu

Last Time: CTFS

Express a periodic signal as a sum of harmonically related complex exponentials:

\[ x(t) = x(t + T) = \sum_{k=\infty}^{\infty} X[k] e^{\frac{j2\pi kt}{T}} \]

\[ X[k] = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-j\frac{2\pi kt}{T}} dt \]

Today: DTFS

Today, we'll apply those same ideas to DT signals.
**DT Sinusoids**

In general, a DT sinusoid has the form:

\[ x[n] = A \cos(\Omega n + \phi) \]

- \( A \) is referred to as the amplitude
- \( \Omega \) is referred to as the *discrete frequency*
- \( \phi \) is referred to as the phase offset

Importantly, \( n \) is always an integer!

---

**DT Sinusoids: Aliasing**

Because \( n \) is an integer, we can only faithfully represent frequencies in the **base band**: \( 0 \leq \Omega \leq \pi \).

Frequencies outside that range alias to frequencies in that range.

For example, the following graphs are of cosines with \( \Omega = 0.2\pi, 2.2\pi, \) and \( 4.2\pi \), respectively:

They all produce exactly the same samples!

Today: Fourier analysis on DT signals, including consequences of aliasing.

---

**Discrete-time Fourier Series**

DTFS: represent periodic DT signal as a sum of weighted harmonics of a fundamental frequency \( \Omega_0 = \frac{2\pi}{N} \):

\[ x[n] = x[n + N] = \sum_k X[k] e^{j2\pi kn/N} \]

Can solve for \( X[k] \) in a similar fashion to CTFS: multiply by basis function and sum over one period:

\[ X[k] = \frac{1}{N} \sum_{n=N_0}^{n+N-1} x[n] e^{-j2\pi kn/N} \]

These are very similar to the CTFS equations, but they differ in important ways!
Finitely-many Unique Harmonics

A crucial observation is that, for a given fundamental frequency $\frac{2\pi}{N}$, there are only $N$ unique harmonics!

Consider $y[n] = e^{j\Omega n}$, which is periodic in $N$. Given its periodicity in $N$, we can say that:

$$y[n] = e^{j\Omega n} = y[n + N] = e^{j\Omega n} e^{j\Omega N}$$

Thus, $e^{j\Omega N} = 1$, and so $\Omega$ must be one of the $N$th roots of 1.

Example: $N = 8$:

$$e^{j\Omega N} = 1$$

Discrete-time Fourier Series

A periodic DT signal with $N$ samples produces a periodic sequence of $N$ Fourier series coefficients.

$$X[k + N] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k n}{N}}$$

$$x[n] = x[n + N] = \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi k n}{N}}$$

DTFS has just $N$ coefficients, whereas CTFS had infinitely many!
Example
What are the Fourier Series coefficients of the following signal?
\[ x[n] = 0.5 \]

Example
What are the Fourier Series coefficients of the following signal?
\[ x[n] = 1 + \sin \left( \frac{2\pi}{10} n \right) \]

Example
What are the Fourier Series coefficients of the following signal?
\[ x[n] = \frac{5}{2} + 3 \cos \left( \frac{2\pi}{5} n \right) - \frac{3}{2} \sin \left( \frac{2\pi}{4} n \right) \]
Example

Consider a family of signals $x_m[n]$, all periodic in $N = 30$, where

$$x_m[n] = \cos\left(\frac{2\pi mn}{30}\right)$$

How do $X_1[k], X_2[k], \ldots, X_{14}[k]$ compare?
(Imagine plotting them vs $k$)

How do $x_1[n], x_2[n], \ldots, x_14[n]$ compare?
(Imagine plotting them vs $n$)

Example

Consider a family of signals $x_m[n]$, all periodic in $N = 30$, where

$$x_m[n] = \cos\left(\frac{2\pi mn}{30}\right)$$

How do $X_{12}[k]$ and $X_{28}[k]$ compare?
(Imagine plotting them vs $k$)

How do $x_{12}[n]$ and $x_{28}[n]$ compare?
(Imagine plotting them vs $n$)

Example

Consider a family of signals $x_m[n]$, all periodic in $N = 30$, where

$$x_m[n] = \cos\left(\frac{2\pi mn}{30}\right)$$

What is $x_{15}[n]$?
What is $X_{15}[k]$?
Properties of DTFS

Operations on the time-domain representation can often be interpreted as equivalent operations on the FSC. For example, let \( x[n] = Ax_1[n] + Bx_2[n] \), where \( x_1[n] = x_1[n+N] \) and \( x_2[n] = x_2[n+N] \). Then,

\[
X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi mn}{N}} = \frac{1}{N} \sum_{n=0}^{N-1} (Ax_1[n] + Bx_2[n]) e^{-j \frac{2\pi mn}{N}} = A X_1[k] + B X_2[k]
\]

Let \( l = n-m \), then:

\[
Y[k] = \frac{1}{N} \sum_{l=-N}^{N-1} y[l] e^{-j \frac{2\pi kl}{N}} = e^{-j \frac{2\pi Km}{N}} \frac{1}{N} \sum_{l=-N}^{N-1} x[l] e^{-j \frac{2\pi kl}{N}} = e^{-j \frac{2\pi Km}{N}} X[k]
\]

Properties of DTFS

Shifting in time changes phase of FSC. For example, let \( y[n] = x[n-m] \), where \( x[n] = x[n+N] \). Then,

\[
Y[k] = \frac{1}{N} \sum_{n=0}^{N-1} y[n] e^{-j \frac{2\pi mn}{N}} = \frac{1}{N} \sum_{n=0}^{N-1} x[n-m] e^{-j \frac{2\pi mn}{N}}
\]

Let \( m = -n \), then:

\[
Y[k] = \frac{1}{N} \sum_{m=0}^{N-1} x[m] e^{-j \frac{2\pi km}{N}} = \frac{1}{N} \sum_{m=0}^{N-1} x[m] e^{-j \frac{2\pi km}{N}} = X[-k]
\]
Properties of DTFS

If \( x[n] \) is real-valued, \( X[k] = X^*[−k] \).

\[
X[k] = \frac{1}{N} \sum_{n=n_0}^{N+n_0-1} x[n] e^{-j\frac{2\pi kn}{N}}
\]

\[
X[−k] = \frac{1}{N} \sum_{n=n_0}^{N+n_0-1} x[n] e^{j\frac{2\pi kn}{N}} = X^*[k]
\]

Properties of DTFS

For the even and odd parts of a signal, \( x[n] = x_e[n] + x_o[n] \).

\[
x_e[n] = \frac{1}{2} (x[n] + x[−n]) \quad DTFS \quad \frac{1}{2} (X[k] + X^*[k]) = \text{Re} (X[k])
\]

\[
x_o[n] = \frac{1}{2} (x[n] − x[−n]) \quad DTFS \quad \frac{1}{2} (X[k] − X^*[k]) = j\text{Im} (X[k])
\]

This Week

**Today’s Lecture:**
DTFS (and consequences of DT)

**Today’s Recitation:**
Use DTFS to understand and auditory perception

**Next Week:**
Applying Fourier analysis to aperiodic signals