Last Time: CTFS

Express a periodic signal as a sum of harmonically related complex exponentials:

\[ x(t) = x(t + T) = \sum_{k \in \mathbb{N}} X[k] e^{j \frac{2\pi kt}{T}} \]

\[ X[k] = \frac{1}{T} \int_{t_0}^{t_0 + T} x(t) e^{-j \frac{2\pi kt}{T}} dt \]

Today: DTFS

Today, we'll apply those same ideas to DT signals.
DT Sinusoids

In general, a DT sinusoid has the form:

\[ x[n] = A \cos(\Omega n + \phi) \]

- \( A \) is referred to as the amplitude
- \( \Omega \) is referred to as the discrete frequency
- \( \phi \) is referred to as the phase offset

Importantly, \( n \) is always an integer!

DT Sinusoids: Aliasing

Because \( n \) is an integer, we can only faithfully represent frequencies in the base band: \( 0 \leq \Omega \leq \pi \).

Frequencies outside that range alias to frequencies in that range.

For example, the following graphs are of cosines with \( \Omega = 0.2\pi, 2.2\pi, \) and \( 4.2\pi \), respectively:

They all produce exactly the same samples!

Today: Fourier analysis on DT signals, including consequences of aliasing.

Discrete-time Fourier Series

DTFS: represent periodic DT signal as a sum of weighted harmonics of a fundamental frequency \( \Omega_0 = \frac{2\pi}{N} \):

\[ x[n] = x[n + N] = \sum_k X[k] e^{j2\pi kn/N} \]

Can solve for \( X[k] \) in a similar fashion to CTFS: multiply by basis function and sum over one period:

\[ X[k] = \frac{1}{N} \sum_{n=N_0}^{N+N_0-1} x[n] e^{-j2\pi kn/N} \]

These are very similar to the CTFS equations, but they differ in important ways!
Finitely-many Unique Harmonics

A crucial observation is that, for a given fundamental frequency $\frac{2\pi}{N}$, there are only $N$ unique harmonics!

Consider $y[n] = e^{j\Omega n}$, which is periodic in $N$. Given its periodicity in $N$, we can say that:

$$y[n] = e^{j\Omega n} = y[n + N] = e^{j\Omega n}e^{j\Omega N}$$

Thus, $e^{j\Omega N} = 1$, and so $\Omega$ must be one of the $N^{th}$ roots of 1.

Example: $N = 8$:

Discrete-time Fourier Series

A periodic DT signal with $N$ samples produces a periodic sequence of $N$ Fourier series coefficients.

$$X[k] = X[k + N] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}$$

$$x[n] = x[n + N] = \sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi kn}{N}}$$

DTFS has just $N$ coefficients, whereas CTFS had infinitely many!
Example

What are the Fourier Series coefficients of the following signal?

\[ x[n] = 0.5 \]

Example

What are the Fourier Series coefficients of the following signal?

\[ x[n] = 1 + \sin\left(\frac{2\pi}{10}n\right) \]

Example

What are the Fourier Series coefficients of the following signal?

\[ x[n] = \frac{5}{2} + 3 \cos\left(\frac{2\pi}{5}n\right) - \frac{3}{2} \sin\left(\frac{2\pi}{4}n\right) \]
Example

Consider a family of signals \( x_m[\cdot] \), all periodic in \( N = 30 \), where

\[
x_m[n] = \cos \left( \frac{2\pi mn}{30} \right)
\]

How do \( X_1[\cdot], X_2[\cdot], \ldots, X_{14}[\cdot] \) compare?
(Imagine plotting them vs \( k \))

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Consider a family of signals \( x_m[\cdot] \), all periodic in \( N = 30 \), where

\[
x_m[n] = \cos \left( \frac{2\pi mn}{30} \right)
\]

How do \( X_2[\cdot] \) and \( X_{28}[\cdot] \) compare?
(Imagine plotting them vs \( k \))

How do \( x_2[\cdot] \) and \( x_{28}[\cdot] \) compare?
(Imagine plotting them vs \( n \))

Example

Consider a family of signals \( x_m[\cdot] \), all periodic in \( N = 30 \), where

\[
x_m[n] = \cos \left( \frac{2\pi mn}{30} \right)
\]

What is \( x_{15}[\cdot] \)?
What is \( X_{15}[\cdot] \)?
Properties of DTFS

Operations on the time-domain representation can often be interpreted as equivalent operations on the FSC.

For example, let \( x[n] = Ax_1[n] + Bx_2[n] \), where \( x_1[n] = x_1[n + N] \) and \( x_2[n] = x_2[n + N] \). Then,

\[
X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}} = \frac{1}{N} \sum_{n=0}^{N-1} (Ax_1[n] + Bx_2[n])e^{-j\frac{2\pi kn}{N}}
\]

\[
= A \frac{1}{N} \sum_{n=0}^{N-1} x_1[n]e^{-j\frac{2\pi kn}{N}} + B \frac{1}{N} \sum_{n=0}^{N-1} x_2[n]e^{-j\frac{2\pi kn}{N}}
\]

\[
= AX_1[k] + BX_2[k]
\]

Shifting in time changes phase of FSC.

For example, let \( y[n] = x[n - m] \), where \( x[n] = x[n + N] \). Then,

\[
Y[k] = \frac{1}{N} \sum_{n=0}^{N-1} y[n]e^{-j\frac{2\pi kn}{N}} = \frac{1}{N} \sum_{n=0}^{N-1} x[n - m]e^{-j\frac{2\pi kn}{N}}
\]

Let \( l = n - m \), then:

\[
Y[k] = \frac{1}{N} \sum_{l=-m}^{N-m-1} x[l]e^{-j\frac{2\pi l(k+n)}{N}}
\]

\[
= e^{-j\frac{2\pi km}{N}} \frac{1}{N} \sum_{l=-m}^{N-m-1} x[l]e^{-j\frac{2\pi l}{N}}
\]

\[
= e^{-j\frac{2\pi km}{N}} X[k]
\]

Flipping in time flips in frequency.

For example, let \( y[n] = x[-n] \) where \( x[n] = x[n + N] \). Then,

\[
Y[k] = \frac{1}{N} \sum_{n=0}^{N-1} y[n]e^{-j\frac{2\pi kn}{N}} = \frac{1}{N} \sum_{n=0}^{N-1} x[-n]e^{-j\frac{2\pi kn}{N}}
\]

Let \( m = -n \), then:

\[
Y[k] = \frac{1}{N} \sum_{m=-N}^{N-1} x[m]e^{-j\frac{2\pi kn}{N}}
\]

\[
= \frac{1}{N} \sum_{m=-N}^{N-1} x[m]e^{-j\frac{2\pi km}{N}}
\]

\[
= X[-k]
\]
Properties of DTFS

If $x[n]$ is real-valued, $X[k] = X^*[−k]$.

$$X[k] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n] e^{-j \frac{2\pi kn}{N}}$$

$$X[−k] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n] e^{j \frac{2\pi kn}{N}} = X^*[k]$$

Notes

Properties of DTFS

FSC for the even and odd parts of a signal, $x[n] = x_e[n] + x_o[n]$.

$$x_e[n] = \frac{1}{2} (x[n] + x[−n]) \quad \text{DTFS} \quad \frac{1}{2} (X[k] + X^*[k]) = \text{Re} (X[k])$$

$$x_o[n] = \frac{1}{2} (x[n] − x[−n]) \quad \text{DTFS} \quad \frac{1}{2} (X[k] − X^*[k]) = j\text{Im} (X[k])$$

Notes

This Week

Today’s Lecture:
DTFS (and consequences of DT)

Today’s Recitation:
Use DTFS to understand and auditory perception

Next Week:
Applying Fourier analysis to aperiodic signals

Notes