6.003
Signal Processing

Week 2, Lecture B:
CT Fourier Series
(Complex Exponential Form)

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Represented arbitrary periodic functions as sums of sines and cosines at harmonically-related frequencies:

\[ f(t) = f(t + T) = c_0 + \sum_{k=1}^{\infty} \left( c_k \cos \left( \frac{2\pi k}{T} t \right) + d_k \sin \left( \frac{2\pi k}{T} t \right) \right) \]
Last time: CTFS, Trig Form

Synthesis:

\[ f(t) = c_0 + \sum_{k=1}^{\infty} \left( c_k \cos \left( \frac{2\pi k}{T} t \right) + d_k \sin \left( \frac{2\pi k}{T} t \right) \right) \]

Analysis:

\[ c_0 = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) \, dt \]

\[ c_k = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos \left( \frac{2\pi k t}{T} \right) \, dt \]

\[ d_k = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin \left( \frac{2\pi k t}{T} \right) \, dt \]
Different “Views” of a Signal

“Time Domain” view: $f(t)$

“Frequency Domain” view:
Different “Views” of a Signal

“Time Domain” view:

“Frequency Domain” view:
Today

- Complex Exponential Form of the CTFS
- Properties of the CTFS
Spectral Representation

\[ f(t) = c_0 + \sum_{k=1}^{\infty} \left( c_k \cos \left( \frac{k2\pi}{T} t \right) + d_k \sin \left( \frac{k2\pi}{T} t \right) \right) \]

Since \( A \sin(\omega t) + B \cos(\omega t) = C \cos(\omega t + \phi) \), we can equivalently say:

\[ f(t) = \sum_{k=0}^{\infty} r_k \cos \left( \frac{k2\pi}{T} t + \phi_k \right) \]

where:

- \( r_k = \sqrt{c_k^2 + d_k^2} \)
- \( \phi_k = \tan^{-1} \left( -\frac{d_k}{c_k} \right) \)
Toward the CE form of CTFS

\[ f(t) = \sum_{k=0}^{\infty} r_k \cos\left( \frac{k2\pi}{T} t + \phi_k \right) \]

Represent using complex exponentials:

\[ f(t) = \sum_{k=0}^{\infty} r_k \left( e^{j\left( \frac{2\pi kt}{T} + \phi_k \right)} + e^{-j\left( \frac{2\pi kt}{T} + \phi_k \right)} \right) \frac{2}{2} \]

Separate complex exponentials and rearrange:

\[ f(t) = \sum_{k=0}^{\infty} \frac{r_k}{2} e^{j\phi_k} e^{j\frac{2\pi kt}{T}} + \sum_{k=0}^{\infty} \frac{r_k}{2} e^{-j\phi_k} e^{-j\frac{2\pi kt}{T}} \]

Let \( A_k = \frac{r_k}{2} e^{j\phi_k} \):

\[ f(t) = \sum_{k=0}^{\infty} A_k e^{j\frac{2\pi kt}{T}} + \sum_{k=0}^{\infty} A_k^* e^{-j\frac{2\pi kt}{T}} \]
Toward the CE form of CTFS

\[ f(t) = \sum_{k=0}^{\infty} A_k e^{j\frac{2\pi k t}{T}} + \sum_{k=0}^{\infty} A_k^* e^{-j\frac{2\pi k t}{T}} \]

Change order of summation on second sum:

\[ f(t) = \sum_{k=0}^{\infty} A_k e^{j\frac{2\pi k t}{T}} + \sum_{k=-\infty}^{0} A_k^* e^{j\frac{2\pi k t}{T}} \]

Let \( F[k] = \begin{cases} A_k & \text{if } k > 0, \\ 2A_0 = c_o & \text{if } k = 0 \\ A_{-k}^* & \text{if } k < 0, \end{cases} \)

\[ f(t) = \sum_{k=-\infty}^{\infty} F[k] e^{j\frac{2\pi k t}{T}} \]
Represent signal $f(\cdot)$ as a sum of harmonically-related complex exponentials:

$$f(t) = \sum_{k=-\infty}^{\infty} F[k] e^{j\frac{2\pi kt}{T}}$$
Determining Coefficients

Similar to determining a coefficient in trig form. Assume

\[ x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} X[k]e^{j\frac{2\pi kt}{T}} \]

To find \( X[m] \), multiply by \( e^{-j\frac{2\pi mt}{T}} \) and integrate:

\[
\int_{T} x(t)e^{-j\frac{2\pi mt}{T}} dt = \int_{T} \sum_{k=-\infty}^{\infty} X[k]e^{j\frac{2\pi kt}{T}} e^{-j\frac{2\pi mt}{T}} dt
\]

\[
= \int_{T} \sum_{k=-\infty}^{\infty} X[k]e^{j\frac{2\pi (k-m)t}{T}} dt
\]

\[
= \sum_{k=-\infty}^{\infty} X[k] \int_{T} e^{j\frac{2\pi (k-m)t}{T}} dt
\]

Terms are equal to \( TX[k] \) if \( k = m \), and 0 otherwise. Therefore:

\[
X[k] = \frac{1}{T} \int_{T} x(t)e^{-j\frac{2\pi kt}{T}} dt
\]
CTFS: Complex Exponential Form

Synthesis:

\[ x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} X[k] e^{j \frac{2\pi kt}{T}} \]

Analysis:

\[ X[k] = \frac{1}{T} \int_{T} x(t)e^{-j \frac{2\pi kt}{T}} dt \]
Complex Exponentials are Your Friends

The complex exponential form of the Fourier series is by far its most frequently-used form. Why?

- Cosine and Sine all-in-one!
  - No need to do separate math for sines and cosines
- No need to memorize trig identities!
- As we’ll see, certain kinds of manipulations are easy to see in the frequency domain when using CE form.
Check Yourself!

What are the Fourier series coefficients (complex exponential form) of the following signals?

\[ x_1(t) = \cos(2\pi t) \]

\[ x_2(t) = \sin(2\pi t) \]

\[ x_3(t) = \cos(2\pi t - \pi/4) \]

**Hint:** We want to express these as sums of complex exponentials. Can you do this without explicitly taking an integral?
Let $X[k]$ represent the Fourier series coefficients of the following signal:

$$x(t) = \frac{1}{2} - \frac{1}{2}u(t)$$

Which of the following is/are true?

1. $X[k] = 0$ if $k$ is even
2. $X[k]$ is always real-valued
3. $|X[k]|$ decreases with $k^2$
4. there are an infinite number of non-zero $X[k]$
Consider $y(t) = Ax_1(t) + Bx_2(t)$, where $x_1$ and $x_2$ are periodic in $T$. What are the CTFS coefficients $Y[k]$, in terms of $X_1[k]$ and $X_2[k]$?
Consider $y(t) = x(t - t_0)$, where $x$ is periodic in $T$. What are the CTFS coefficients $Y[k]$, in terms of $X[k]$?
Consider $y(t) = \frac{d}{dt} x(t)$, where $x$ is periodic in $T$. What are the CTFS coefficients $Y[k]$, in terms of $X[k]$?
Properties of CTFS

Certain operations in the time domain have predictable effects in the frequency domain. For example:

If \( y(t) = Ax_1(t) + Bx_2(t) \), then \( Y[k] = AX_1[k] + BX_2[k] \)

If \( y(t) = x(t - t_0) \), then \( Y[k] = e^{-j \frac{2\pi k t_0}{T}} X[k] \)

If \( y(t) = \dot{x}(t) \), then \( Y[k] = j \frac{2\pi k}{T} X[k] \)

Often, we can find the Fourier coefficients of a new signal without explicitly integrating by taking advantage of these and other properties.
Let $Y[k]$ represent the Fourier series coefficients of the following signal:

$$y(t) = \frac{1}{8} - \frac{1}{8}$$

Which of the following is/are true?

1. $Y[k] = 0$ if $k$ is even
2. $Y[k]$ is real-valued
3. $|Y[k]|$ decreases with $k^2$
4. there are an infinite number of non-zero $Y[k]$

**Hint:** How does this relate to the square wave?
Trig Form: Even and Odd Parts

c_k’s (cosines) alone can only represent even functions.
d_k’s (sines) alone can only represent odd functions.

An arbitrary signal \( x(t) \) can be broken down into even and odd parts:

\[
x(t) = x_e(t) + x_o(t), \text{ where:}
\]

\[
x_e(t) = \frac{x(t) + x(-t)}{2} \quad x_o(t) = \frac{x(t) - x(-t)}{2}
\]

The even part shows up in the \( c_k \) coefficients, and the odd part shows up in the \( d_k \) coefficients.
If \( x(t) = x_e(t) + x_o(t) \), where \( x_e(\cdot) \) is even and \( x_o(\cdot) \) is odd, what are the Fourier coefficients of \( x_e(\cdot) \) and \( x_o(\cdot) \), in terms of \( X[k] \)?

\[
X_e[k] = \ \\
\]

\[
X_o[k] = \ \\
\]
Summary

Today:
- Introduced complex exponential form of CTFS
- Examined properties of the CTFS

Recitation:
- More practice with CE form of CTFS

Next Week:
- No lecture/recitation on Tuesday
- DTFS on Thursday